

Chapter 8

Newtonian Mechanics

In Chapters 4 and 5 we saw how to use calculus and the computer in order to predict the motion of a projectile. We saw that if we knew the initial position and velocity of an object, and had a formula for its acceleration vector, then we could predict its position far into the future.

To go beyond a discussion of projectile motion, to develop a general scheme for predicting motion, two new concepts are needed. One is mass, discussed in chapter 6, and the other is force, to be introduced now. We will see that once we know the forces acting on an object, we can obtain a formula for the object's acceleration and then use the techniques of Chapters 4 and 5 to predict motion. This scheme was developed in the late 1600s by Isaac Newton and is known as Newtonian Mechanics.

FORCE

The concept of a force—a push or a pull—is not as strange or unfamiliar as the acceleration vector we have been discussing. When you push on an object you are exerting a force on that object. The harder you push, the stronger the force. And the direction you push is the direction of the force. From this we see that force is a quantity that has a magnitude and a direction. As a result, it is reasonable to assume that a force is described mathematically by a vector, which we will usually designate by the letter \vec{F} .

It is often easy to see when forces are acting on an object. What is more subtle is the relationship between force and the resulting acceleration it produces. If I push on a big tree, nothing happens. I can push as hard as I want and the tree does not move. (No bulldozers allowed.) But if I push on a chair, the chair may move. The chair moves if I push sideways but not if I push straight down.

The ancient Greeks, in particular, Aristotle, thought that there was a direct relationship between force and velocity. He thought that the harder you pushed on an object, the faster it went. There is some truth in this if you are talking about pushing a stone along the ground or pulling a boat through water. But these examples, which were familiar problems in ancient time, turn out to be complex situations, involving friction and viscous forces.

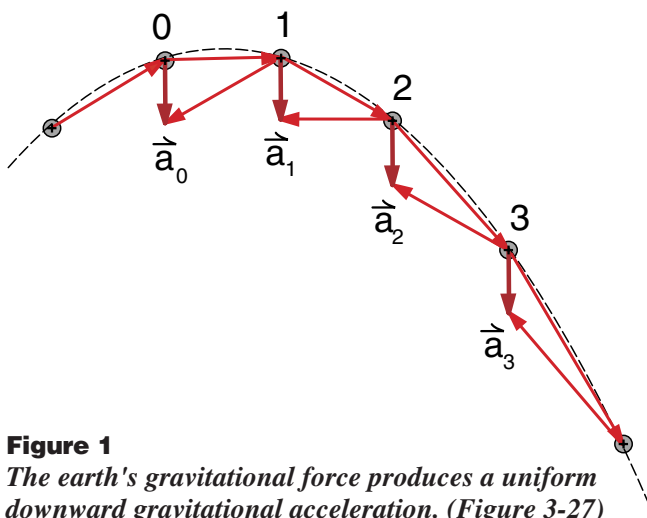


Figure 1
The earth's gravitational force produces a uniform downward gravitational acceleration. (Figure 3-27)

Only when Galileo focused on a problem without much friction – projectile motion – did the important role of the acceleration vector become apparent. Later, Newton compared the motion of a projectile (the apple that supposedly fell on his head) with the motion of the planets and the moon, giving him more examples of motion without friction. These examples led Newton to the discovery that force is directly related to acceleration, not velocity.

In our discussion of projectile motion, and projectile motion with air resistance, we have begun to see the relation between force and acceleration. While a projectile is in flight, and we can neglect air resistance, the projectile's acceleration is straight down, in the direction of the earth as shown in Figure (1). As we stand on the earth, we are being pulled down by gravity. While the projectile is in flight, it is also being pulled down by gravity. It is a reasonable guess that the projectile's downward acceleration vector \vec{g} is caused by the gravitational force of the earth.

When we considered the motion of a particle at constant speed in a circle as shown in Figure (2), we saw that the particle's acceleration vector pointed toward the center of the circle. A simple physical example of this circular motion was demonstrated when we tied a golf ball to a string and swing it over our head.

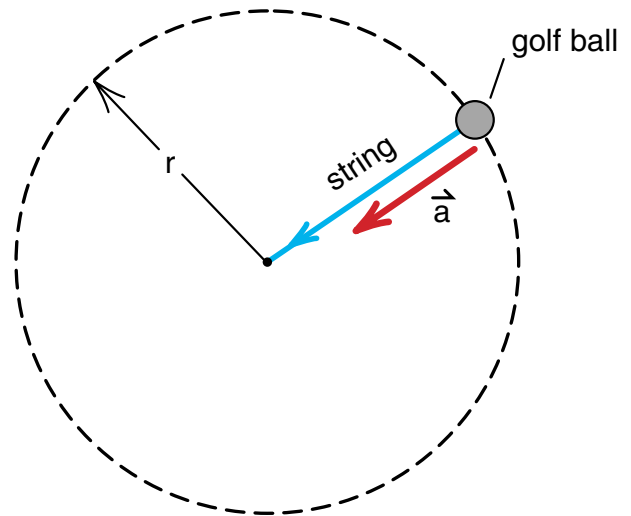


Figure 2
The acceleration of the ball is in the same direction as the force exerted by the string. (Figure 3-28)

While swinging the golf ball, it was the string pulling on the ball that kept the ball moving in a circle. (Let go of the string and the ball goes flying off.) The string is capable of pulling only along the length of the string, which in this case is toward the center of the circle. Thus the force exerted by the string is in the direction of the golf ball's acceleration vector. This makes our second example in which the particle's acceleration vector points in the same direction as the force exerted on it.

The example of projectile motion with air resistance, shown in Figure (3), presented a more complex situation. In our study of the motion of a Styrofoam projectile, we had two forces acting on the ball. There was the downward force of gravity, and also the force exerted by the wind we would feel if we were riding along with the ball. We saw that gravity and the wind each produced an acceleration vector, and that the ball's actual acceleration was the vector sum of the two individual accelerations. This is an important clue as to how we should handle situations where more than one force is acting on an object.

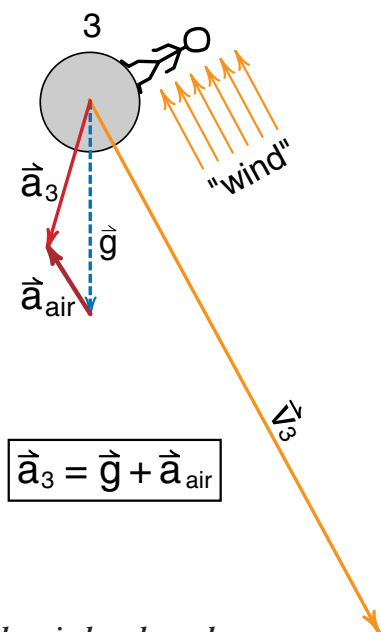


Figure 3
Gravity and the wind each produce an acceleration, \vec{g} and \vec{a}_{air} respectively. The net acceleration of the ball is the vector sum of the two accelerations.

THE ROLE OF MASS

Our three examples, projectile motion, motion in a circle, and projectile motion with air resistance, all demonstrate that a force produces an acceleration in the direction of the force. The next question is—how much acceleration? Clearly not all forces have the same effect. If I shove a child's toy wagon, the wagon might accelerate rapidly and go flying off. The same shove applied to a Buick automobile will not do very much.

There is clearly a difference between a toy wagon and a Buick. The Buick has much more mass than the wagon, and is much less responsive to my shove.

In our recoil definition of mass discussed in Chapter 6 and illustrated in Figure (4), we defined the ratio of two masses as the inverse ratio of their recoil speeds

$$\frac{m_1}{m_2} = \frac{v_2}{v_1}$$

The intuitive idea is that the more massive the object, the slower it recoils. The more mass, the less responsive it is to the shove that pushed the carts apart.

Think about the spring that pushes the cart apart in our recoil experiment. Once we burn the thread holding the carts together, the spring pushes out on both carts, causing them to accelerate outward. If the spring is pushing equally hard on both carts (later we will see that it must), then we see that the resulting acceleration and final velocities are inversely proportional to the mass of the cart. If m_1 is twice as massive as m_2 , it gets only half as much acceleration from the same spring force. Our recoil definition and experiments on mass suggests that the effectiveness of a force in producing an acceleration is inversely proportional to the object's mass. For a given force, if you double the mass, you get only half the acceleration. That is the simplest relationship between force and mass that is consistent with our general experience, and it turns out to be the correct one.

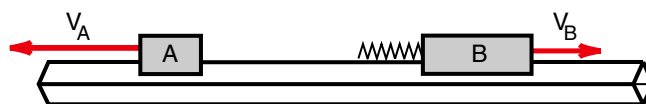


Figure 4
Definition of mass. When two carts recoil from rest, the more massive cart recoils more slowly.

NEWTON'S SECOND LAW

We have seen that a force \vec{F} acting on a mass m , produces an acceleration \vec{a} that 1) is in the direction of \vec{F} , and 2) has a magnitude inversely proportional to m . The simplest equation consistent with these observations is

$$\vec{a} = \frac{\vec{F}}{m} \quad (1)$$

Equation (1) turns out to be the correct relationship, and is known as *Newton's Second Law of Mechanics*. (The *First Law* is a statement of the special case that, if there are no forces, there is no acceleration. That was not obvious in the late 1600s, and was therefore stated as a separate law.) A more familiar form of Newton's second law, seen in all introductory physics texts is

$$\boxed{\vec{F} = m\vec{a}} \quad (1a)$$

If there is any equation that is essentially an icon for the introductory physics course, Equation (1a) is it.

At this point Equation (1) or (1a) serves more as a definition of force than a basic scientific result. We can, for example, see from Equation (1a) that force has the dimensions of mass times acceleration. In the MKS system of units this turns out to be $\text{kg}(\text{m}/\text{sec}^2)$, a collection of units called the *newton*. Thus we can say that we push on an object with a force of so many newtons. In the CGS system, the dimensions of force are $\text{gm}(\text{cm}/\text{sec}^2)$, a set of units called a *dyne*. A dyne turns out to be a very small unit of force, of the order of the force exerted by a fly doing push-ups. The newton is a much more convenient unit. The real confusion is in the English system of units where force is measured in *pounds*, and the unit of mass is a *slug*. We will carefully avoid doing Newton's law calculations in English units so that the student does not have to worry about pounds and slugs.

At a more fundamental level, we can use Equation (1) to *detect the existence of a force by the acceleration it produces*. In projectile motion, how do we know that there is a gravitational force \vec{F}_g acting on the projectile? Because of the gravitational acceleration. The acceleration \vec{a} due to gravity is equal to \vec{g} ($9.8 \text{ m}/\text{sec}^2$ directed downward), thus we can say that the gravitational force \vec{F}_g that produces this acceleration is

$$\boxed{\vec{F}_g = m\vec{g}} \quad \begin{array}{l} \text{gravitational force} \\ \text{on a mass } m \end{array} \quad (2)$$

where m is the mass of the projectile.

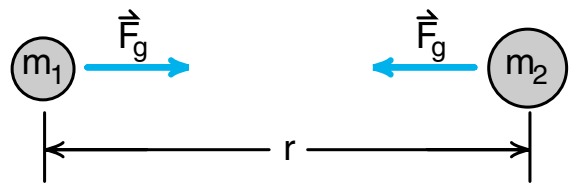


Figure 5
The gravitational force between small masses is proportional to the product of the masses, and inversely proportional to the square of the separation between them.

NEWTON'S LAW OF GRAVITY

Newton went beyond using the second law to define force; he also discovered a basic law for the gravitational force between objects. With Newton's law of gravity combined with Newton's second law, we can make detailed predictions about how projectiles, satellites, planets, and solar systems behave. This combination, where one has an explicit formula for gravitational forces, and the second law to predict what accelerations these forces produce, was one of the most revolutionary scientific discoveries ever made.

Newton's so-called *universal law of gravitation* can most simply be stated as follows. If we have two small masses of mass m_1 and m_2 , separated by a distance r as shown in Figure (5), then the force between them is proportional to the product $m_1 m_2$ of their masses, and inversely proportional to the square of the distance r between them. This can be written as an equation of the form

$$\boxed{|\vec{F}_g| = G \frac{m_1 m_2}{r^2}} \quad \text{Newton's law of gravity} \quad (3)$$

where the proportionality constant G is a number that must be determined by experiment.

Equation (3) itself is not the whole story, we must make several more points. First, and very important, is the fact that gravitational forces are always attractive; m_1 is pulled directly toward m_2 , and m_2 directly toward m_1 . Second, the strength of these forces are equal, even if m_2 is much bigger than m_1 , the force of m_2 on m_1 is the same in strength as the force of m_1 on m_2 . That is why we used the same symbol \vec{F}_g for the two attractive forces in Figure (5).

Newton's law of gravity is called the *universal law of gravitation* because Equation (3) is supposed to apply to all masses anywhere in the universe, with the same numerical constant G everywhere. G is called the *universal gravitational constant*, and has the numerical value, in the MKS system of units

$$G = 6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg sec}^2} \quad \text{universal gravitational constant} \quad (4)$$

We will discuss shortly how this number was first measured.

Exercise 1

Combine Newton's second law $\vec{F} = m\vec{a}$ with the law of gravity $|\vec{F}_g| = Gm_1 m_2 / r^2$ and show that the dimensions for G in Equation (4) are correct.

Big Objects

In our statement of Newton's law of gravity, we were careful to say that Equation (3) applied to two small objects. To be more explicit, we mean that the two objects m_1 and m_2 should be small in dimensions compared to the separation r between them. We can think of Equation (3) as applying to two *point particles* or *point masses*.

What happens if one or both of the objects are large compared to their separation? Suppose, for example, that you would like to calculate the gravitational force between you and the earth as you stand on the surface of the earth. The correct way to do this is to realize that you are attracted, gravitationally, to every rock, tree, every single piece of matter in the entire earth as indicated in Figure (6). Each of these pieces of matter is pulling on you, and together they produce a net gravitational force \vec{F}_g which is the force $m\vec{g}$ that we saw in our discussion of projectile motion.

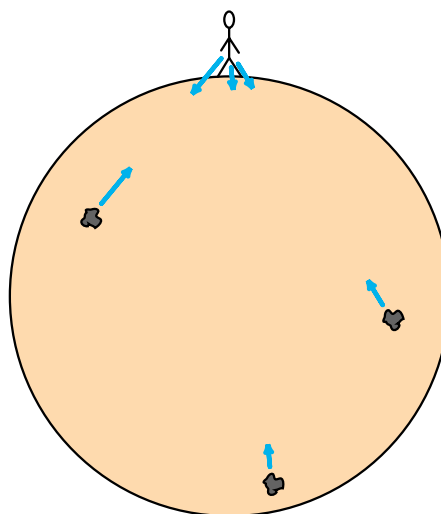


Figure 6
You are attracted to every piece of matter in the earth.

It appears difficult to add up all the individual forces exerted by every chunk of matter in the entire earth, to get the net force \vec{F}_g . Newton also thought that this was difficult, and according to some historical accounts, invented calculus to solve the problem. Even with calculus, it is a fairly complicated problem to add up all of these forces, but the result turns out to be very simple. **For any uniformly spherical object, you get the correct answer in Newton's law of gravity if you think of all the mass as being concentrated at a point at the center of the sphere.** (This result is an accidental consequence of the fact that gravity is a $1/r^2$ force, i.e., that it is inversely proportional to the square of the distance. We will have much more to say about this accident in later chapters.)

Since the earth is nearly a uniformly spherical object, you can calculate the gravitational force between you and the earth by treating the earth as a point mass located at its center, 4000 miles below you, as indicated in Figure (7).

Galileo's Observation

As we mentioned earlier, Galileo observed that, in the absence of air resistance, all projectiles should have the same acceleration no matter what their mass. This leads to the striking result that, in a vacuum, a steel ball and a feather fall at the same rate. Now we can see that this is a consequence of Newton's second law combined with Newton's law of gravity.

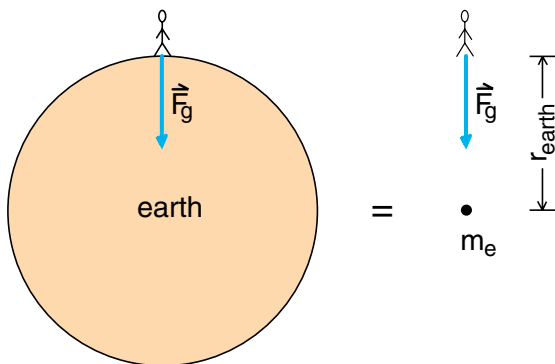


Figure 7
The gravitational force of the entire earth acting on you is the same as the force of a point particle with a mass equal to the earth mass, located at the earth's center, one earth radius below you.

Using the results of Figure (7), i.e., calculating \vec{F}_g by replacing the earth by a point mass m_e located a distance r_e below us, we get

$$F_g = \frac{Gmm_e}{r_e^2} \tag{5}$$

for the strength of the gravitational force on a particle of mass m at the surface of the earth. Combining this with Newton's second law

$$\vec{F}_g = m\vec{g} \text{ or } F_g = mg \tag{6}$$

we get

$$mg = \frac{Gmm_e}{r^2} \tag{7}$$

The important result is that the particle's mass m cancels out of Equation (7), and we are left with the formula

$$g = \frac{Gm_e}{r_e^2} \tag{8}$$

for the acceleration due to gravity. We note that g depends on the earth mass m_e , the earth radius r_e , and the universal constant G , but **not on the particle's mass m** . Thus objects of different mass should have the same acceleration.

Galileo Was Right!

By WILLIAM HINES

HOUSTON — Galileo was right and Apollo 15 Astronaut David R. Scott can prove it.

The 17-th century Italian mathematician believed contrary to then prevailing scientific opinion that gravity exerts an equal influence on all things, regardless of their size, shape or weight.

He even went so far as to say that if it weren't for the resistance offered by air, a cannonball and a feather would fall at the same speed.

It wasn't possible for Galileo to demonstrate the truth of that assertion, although legend says he performed a compromise experiment by dropping a large iron ball and a small one off the leaning tower of Pisa and that

they hit the ground at the same instant.

Just before getting back into the Lunar Landing Craft Falcon fc1 takeoff from the Moon, Scott demonstrated that Galileo was right.

Holding a metal geological hammer in his right hand and a feather ("A falcon feather," he explained) in his right, Scott faced the Apollo television camera and released the two objects.

Falling slowly in the eak gravity of the airless Moon, the hammer and the feather reached the surface at precisely the same instant.

"How about that?" the delighted Scott remarked. "Mr. Galileo was correct in his findings."

THE CAVENDISH EXPERIMENT

A key feature of Newton's law of gravitation is that all objects attract each other via gravity. Yet in practice, the only gravitational force we ever notice is the force of attraction to the earth. What about the gravitational force between two students sitting beside each other, or between your two fists when you hold them close to each other? The reason that you do not notice these forces is that the gravitational force is incredibly weak, weak compared to other forces that hold you, trees, and rocks together. Gravity is so weak that you would never notice it except for the fact that you are on top of a huge hunk of matter called the earth. The earth mass is so great that, even with the weakness of gravity, the resulting force between you and the earth is big enough to hold you down to the surface.

The gravitational force between two reasonably sized objects is not so small that it cannot be detected, it just requires a very careful experiment that was first performed by Henry Cavendish in 1798. In the Cavendish experiment, two small lead balls are mounted on the end of a light rod. This rod is then suspended on a fine glass fiber as shown in Figure (8a).

As seen in the top view in Figure (8b), two large lead balls are placed near the small ones in such a way that the gravitational force between each pair of large and small balls will cause the rod to rotate in one direction. Once the rod has settled down, the large lead balls are moved to the position shown in Figure (8c). Now the gravitational force causes the rod to rotate the other way. By measuring the angle that the rod rotates, and by measuring what force is required to rotate the rod by this angle, one can experimentally determine the strength of the gravitational force \vec{F}_g between the balls. Then by using Newton's law of gravity

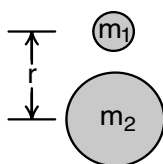
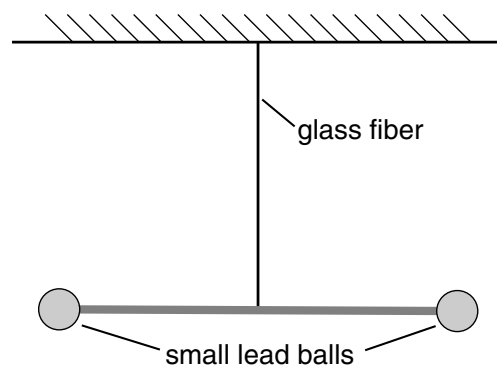
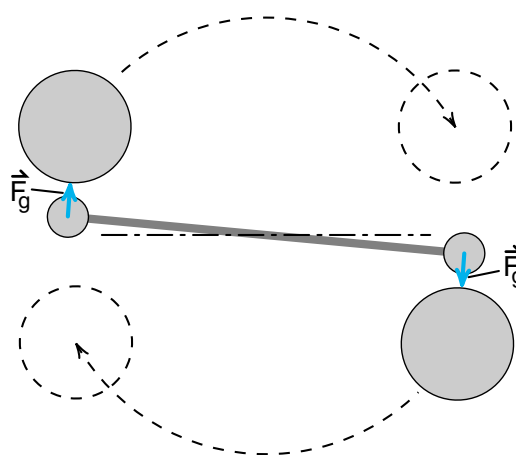
$$F_g = G \frac{m_1 m_2}{r^2}$$


Figure 9

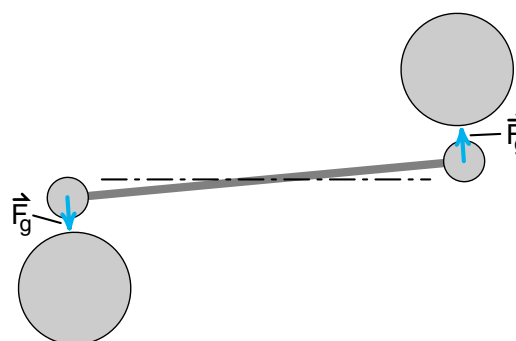
applied to Figure (9), one can solve for G in terms of the known quantities F_g , m_1 , m_2 and r^2 . This was the way that Newton's universal constant G , given in Equation (4) was first measured.



a) Side view of the small balls.



b) Top view showing two large lead balls.



c) Top view with large balls rotated to new position.

Figure 8

The Cavendish experiment. By moving the large lead balls, the small lead balls are first pulled one way, then the other. By measuring the angle the stick holding the small balls is rotated, one can determine the gravitational force \vec{F}_g .

"Weighing" the Earth

Once you know G , you can go back to the formula (8) for the acceleration g due to gravity, and solve for the earth mass m_e to get

$$m_e = \frac{gr_e^2}{G} = \frac{9.8 \text{ m/sec}^2 \times (6.37 \times 10^6 \text{ m})^2}{6.67 \times 10^{-11} \text{ m}^3/\text{kg sec}^2}$$

$$= 6.0 \times 10^{24} \text{ kg} \quad (9)$$

As a result, Cavendish was able to use his value for G to determine the mass of the earth. This was the first determination of the earth's mass, and as a result the Cavendish experiment became known as the experiment that "weighed the earth".

Exercise 2

The density of water is 1 gram/cm³. The average density of the earth's outer crust is about 3 times as great. Use Cavendish's result for the mass of the earth to decide if the entire earth is like the crust. (Hint—the volume of a sphere of radius r is $4/3\pi r^3$). Relate your result to what you have read about the interior of the earth.

Inertial and Gravitational Mass

The fact that, in the absence of air resistance, all projectiles have the same acceleration—the fact that the m 's canceled in Equation (7), has a deeper consequence than mere coincidence. In Newton's second law, the m in the formula $\vec{F} = m\vec{a}$ is the mass defined by the recoil definition of mass discussed in Chapter 6. Called *inertial mass*, it is the concept of mass that we get from the law of conservation of linear momentum.

In Newton's law of gravity, the projectile's mass m in the formula $F_g = Gmm_e/r_e^2$ is what we should call the *gravitational mass* for it is defined by the gravitational interaction. It is the experimental observation that the m 's cancel, the observation that all projectiles have the same acceleration due to gravity, that tells us that the inertial mass is the same as gravitational mass. This equivalence of inertial and gravitational mass has been tested with extreme precision to one part in a billion by Etvös in 1922 and to even greater accuracy by R. H. Dicke in the 1960s.

SATELLITE MOTION

The key idea that led Newton to his universal law of gravitation was that the moon, while traveling in its orbit about the earth, was subject to the same kind of force as an apple falling from a tree. We have seen that a projectile in flight, such as an apple, accelerates down toward the center of the earth. The moon, in its nearly circular orbit around the earth, also accelerates toward the center of the earth, as illustrated in Figure (10). Newton proposed that the accelerations of the falling apple and of the orbiting moon were both caused by the gravitational pull of the earth.

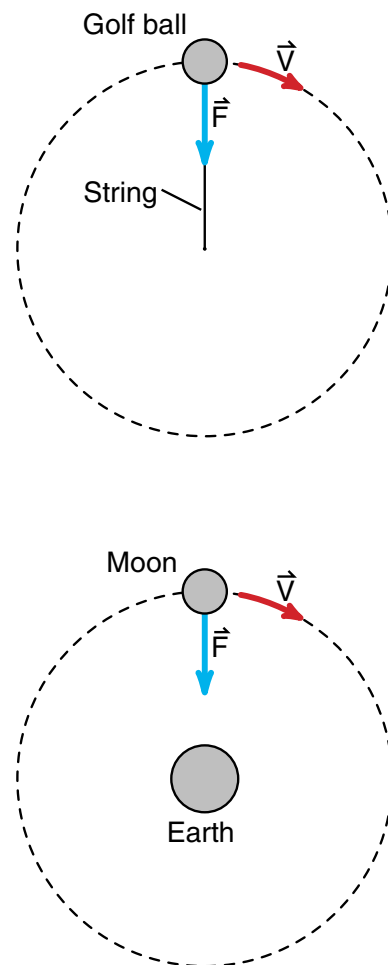


Figure 10

When we swing a golf ball in a circle, the ball accelerates toward the center of the circle, in the direction it is pulled by the string. Similarly, the moon, in its circular orbit about the earth, accelerates toward the center of the earth, in the direction it is pulled by the earth's gravity.

The moon, being farther away from the center of the earth should be expected to feel a weaker gravitational force and therefore have a weaker acceleration. From direct calculation Newton could determine how much weaker the moon's acceleration was, and thus determine how the gravitational acceleration and force decreases with distance.

To repeat Newton's calculation, we know that the apple on the surface of the earth has an acceleration $g_{\text{apple}} = 9.8 \text{ m/sec}^2$. To determine the magnitude of the moon's orbital acceleration toward the earth, $g_{\text{moon orbit}}$, we can use the formula derived in Chapter 3 for uniform circular motion, namely

$$|\vec{a}| = g_{\text{moon orbit}} = \frac{v^2}{r} \quad \text{uniform circular motion} \quad (3-12)$$

To calculate the speed v of the moon, we note that the moon takes 27.32 days or 2.36×10^6 seconds for one complete orbit. The radius of the moon orbit is 3.82×10^8 meters, so that

$$\begin{aligned} v_{\text{moon}} &= \frac{\text{orbital circumference}}{\text{time for one orbit}} = \frac{2\pi r}{t_{\text{orbit}}} \\ &= \frac{2\pi \times (3.82 \times 10^8 \text{ meters})}{2.36 \times 10^6 \text{ sec}} \\ &= 1.02 \times 10^3 \frac{\text{m}}{\text{sec}} \end{aligned} \quad (10)$$

or very close to 1 kilometer per second. Substituting this value of v into the formula v^2/r , gives

$$\begin{aligned} g_{\text{moon orbit}} &= \frac{\left(1.02 \times 10^3 \text{ m/sec}\right)^2}{3.82 \times 10^8 \text{ m}} \\ &= 2.70 \times 10^{-3} \frac{\text{m}}{\text{sec}^2} \end{aligned} \quad (11)$$

The ratio of the moon's orbital acceleration to the apple's acceleration

$$\begin{aligned} \frac{g_{\text{moon orbit}}}{g_{\text{apple}}} &= \frac{2.70 \times 10^{-3} \text{ m/sec}^2}{9.8 \text{ m/sec}^2} \\ &= 2.71 \times 10^{-4} \end{aligned} \quad (12)$$

I.e., the moon's acceleration is 27 thousand times weaker than the apple's.

To understand the meaning of this result, let us look at the square of the ratio of the distances from the apple to the center of the earth, and the moon to the center of the earth. We have

$$\begin{aligned} \left(\frac{r_{\text{apple to center of earth}}}{r_{\text{moon orbit}}}\right)^2 &= \left(\frac{6.37 \times 10^6 \text{ m}}{3.82 \times 10^8 \text{ m}}\right)^2 \\ &= 2.78 \times 10^{-4} \end{aligned} \quad (13)$$

which, to the accuracy of our work, is the same as the ratio of accelerations.

Equating the results in Equations (12) and (13), we get

$$\begin{aligned} \frac{g_{\text{moon orbit}}}{g_{\text{apple}}} &= \frac{r_e^2}{r_{\text{moon orbit}}^2} \\ g_{\text{moon orbit}} &= \frac{g_{\text{apple}} \times r_e^2}{r_{\text{moon orbit}}^2} \propto \frac{1}{r_{\text{moon orbit}}^2} \end{aligned} \quad (14)$$

Where $g_{\text{apple}} \times r_e^2$ can be thought of as a constant.

From such calculations Newton saw that the gravitational acceleration of the moon, and thus the gravitational force, decreased as the square of the distance from the moon to the center of the earth. This was how Newton deduced that gravity was a $1/r^2$ force law.

Exercise 3

How far above the surface of the earth do you have to be so that, in free fall, your acceleration is half that of objects near the surface of the earth?

Other Satellites

To explain to the world the similarity of projectile and satellite motion, that both the apple and the moon were simply falling toward the center of the earth, Newton drew the sketch shown in Figure (11). In the sketch, Newton shows a projectile being fired horizontally from the top of a mountain, and shows what would happen if there were no air resistance. If the horizontal velocity were not too great, the projectile would go a short distance along the typical parabolic path we have studied in the strobe labs. As the projectile is fired faster it would travel farther before hitting the ground. Finally we reach a point where the projectile keeps falling toward the earth, but the earth keeps falling away and the projectile goes all the way around the earth without hitting it.

Another perspective of the same idea is illustrated in Figures (12) and (13). Figure (12) is a strobe photograph showing two steel balls launched simultaneously, one being dropped straight down and the other being fired horizontally. The photograph clearly demonstrates that the downward motion of the two projectiles

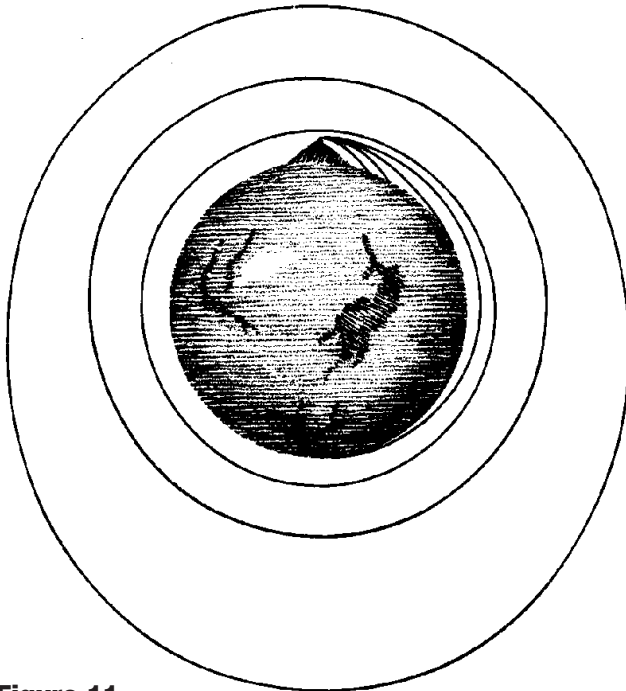


Figure 11
Newton's sketch showing that the difference between projectile and satellite motion is that satellites travel farther. Both are accelerating toward the center of the earth.

is the same. By using the constant acceleration formulas with $g = 32 \text{ ft/sec}^2$, we can easily calculate that at the end of one second both projectiles will have fallen 16 ft, and at the end of two seconds a distance of 64 ft.

In Figure (13), we have sketched the curved surface of the earth. Due to this curvature, the surface of the earth will be 16 ft below a horizontal line out at a distance of 4.9 miles, and 64 ft below at a distance of 9.8 miles. This effect can be seen from a small boat as you leave shore. When you are 10 miles off shore, you cannot see lighthouses under 64 ft tall, unless you climb your own mast. (For landlubbers sunning on the beach, sailboats with 64 ft high masts disappear from sight at a distance of 10 miles.)

Comparing Figures (12) and (13), we see that in the absence of air resistance, if a projectile were fired horizontally at a speed of 4.9 miles per second, during the first second it would fall 16 ft, but the earth would have also fallen 16 ft, and the projectile would be no closer to the surface. By the end of the 2nd second the projectile would have fallen 64 ft, but still not have come any closer to the surface of the earth. Such a projectile would keep traveling around the earth, never hitting the surface. It would fall all the way around, becoming an earth satellite.

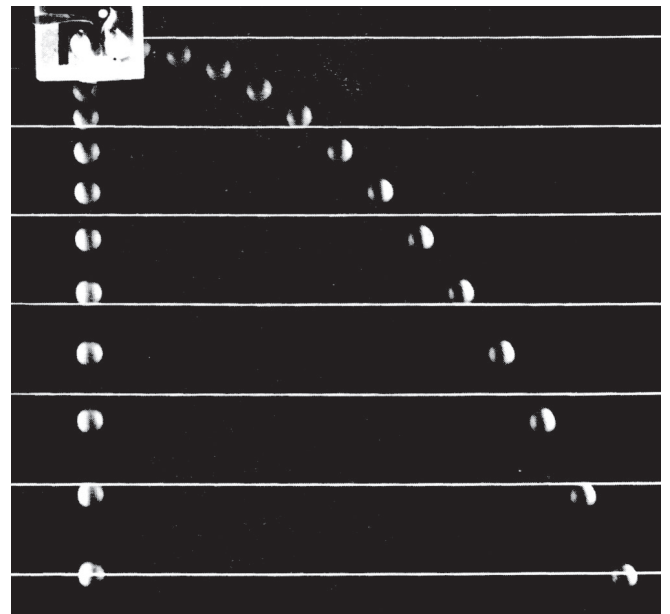


Figure 12
Two projectiles, released simultaneously. The horizontal motion has no effect on the vertical motion: they both fall at the same rate.

Exercise 4

An earth satellite in a low orbit, for instance 100 miles up, is so close to the surface of the earth (100 miles is so small compared to the earth's radius of 4000 miles) that the satellite's acceleration is essentially the same as the acceleration of projectiles here on earth. Use this result to predict the period T of the satellite's orbit. (Hint – the satellite travels one earth circumference $2\pi r_e$ in one period T . This allows you to calculate the satellite's speed v . You then use the formula v^2/r for the magnitude of the satellite's acceleration.)

Weight

The popular press often talks about the astronauts in spacecraft orbiting the earth as being *weightless*. This is verified by watching them on television floating around inside the space capsule. You might jump to the conclusion that because the astronauts are floating around in the capsule, they do not feel the effects of gravity. This is true in the same sense that when you jump off a high diving board, you do not feel the effects of gravity—until you hit the water. While you are falling, you are *weightless* just like the astronauts.

The only significant difference between your fall from the high diving board, and the astronaut's weightless experience in the space capsule, is that the astronaut's experience lasts longer. As the space capsule orbits the earth, the capsule and the astronauts inside are in continuous free fall. They have not escaped the earth's gravity, it is gravity that keeps them in orbit, accelerating toward the center of the earth. But because they are in free fall, they do not feel the acceleration, and are considered to be *weightless*.

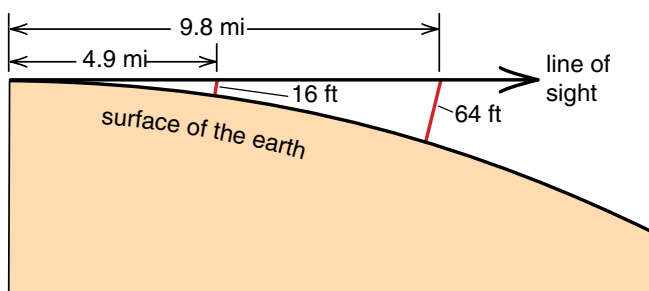


Figure 13

The curvature of the earth causes the horizon to fall away 64 feet at a distance of 9.8 miles.

If the astronaut in an orbiting space capsule is weightless, but still subject to the gravitational force of the earth, we cannot directly associate the word *weight* with the effects of gravity. In order to come up with a definition of the word weight that has some scientific value, and is reasonably consistent with the use of the word in the popular press, we can define the weight of an object as the magnitude of the force the object exerts on the bathroom scales. Here on earth, if you have an object of mass m and you set it on the bathroom scales, it will exert a downward gravitational force of magnitude

$$F_g = mg$$

Thus we say that the object has a weight W given by

$$W = mg \quad (15)$$

For example, a 60 kg boy standing on the scales exerts a gravitational force

$$\begin{aligned} W(60 \text{ kg boy}) &= 60 \text{ kg} \times 9.8 \frac{\text{m}}{\text{sec}^2} \\ &= 588 \text{ newtons} \end{aligned}$$

We see that weight has the dimensions of a force, which in the MKS system is newtons. If the same boy stood on the same scales in an orbiting spacecraft, both the boy and the scales would be in free fall toward the center of the earth, the boy would exert no force on the scales, and he would therefore be weightless.

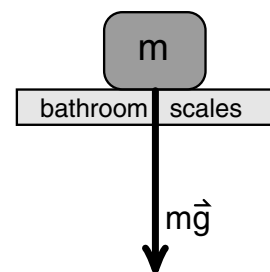


Figure 14

We will define the weight of an object as the force it exerts on the bathroom scales.

Although we try to make the definition of the word *weight* consistent with the popular use of the word, we do not actually succeed. In almost any country except the United States, when you buy a steak, the butcher will weigh it in grams. The grocer will tell you that a banana weighs 200 grams. You are not likely find a grocer who tells you the weight of an object in newtons. It is a universal convention to tell you the mass in grams or kilograms, and say that that is the weight. About the only place will you will find the word *weight* to mean a force, as measured in newtons, is in a physics course.

(In the English system of units, a pound is a force, so that it is correct to say that our 60 kg mass boy weighs 132 lbs. That, of course, leaves us with the question of what mass is in the English units. From the formula $F = mg$, we see that $m = F/g$, or an object that weighs 32 lbs has a mass $32 \text{ lbs}/32 \text{ ft/sec}^2 = 1$. As we mentioned earlier, this unit mass in the English units is called a *slug*. This is the last time we will mention slugs in this text.)

Earth Tides

An aspect of Newton's law of gravity that we have not said much about is the fact that gravity is a mutual attraction. As we mentioned, two objects of mass m_1 and m_2 separated by a distance r , attract *each other* with a gravitational force of magnitude $|\vec{F}_g| = Gm_1m_2/r^2$. The point we want to emphasize now is that the force on *each* particle has the same strength F_g .

Let us apply this idea to you, here on the surface of the earth. Explicitly, let us assume that you have just jumped off a high diving board as illustrated in Figure (15), and have not yet hit the water. While you are falling, the earth's gravity exerts a downward force \vec{F}_g which produces your downward acceleration \vec{g} .

According to Newton's law of gravity, you are exerting an equal and opposite gravitational force \vec{F}_g on the earth. Why does nobody talk about this upward force you are exerting on the earth? The answer, shown in the following exercise, is that even though you are pulling up on the earth just as hard as the earth is pulling down on you, the earth is so much more massive that your pull has no detectable effect.

Exercise 5

Assume that the person in Figure (15) has a mass of 60 kilograms. The gravitational force he exerts on the earth causes an upward acceleration of the earth a_{earth} . Show that $a_{\text{earth}} = 10^{-22} \text{ m/sec}^2$.

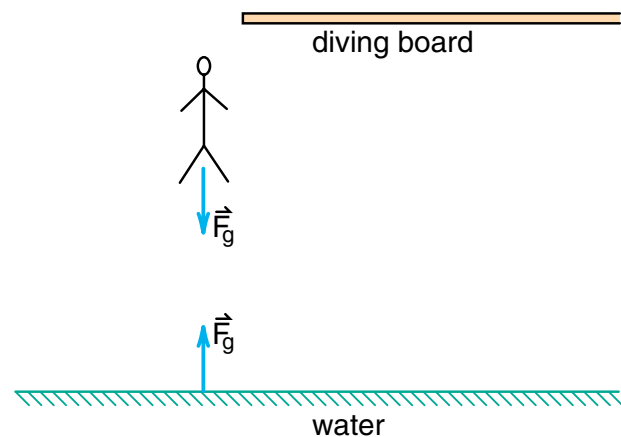


Figure 15

As you fall toward the water, the earth is pulling down on you, and you are pulling up on the earth. The two forces are of equal strength.

More significant than the force of the diver on the earth is the force of the moon on the earth. It is well known that the ocean tides are caused by the moon's gravity acting on the earth. On the night of a full moon, high tide is around midnight when the moon is directly overhead. The time of high tide changes by about an hour a day in order to stay under the moon.

The high tide under the moon is easily explained by the idea that the moon's gravity sucks the ocean water up into a bulge under the moon. As the earth rotates and we pass under the bulge, we see a high tide. This explains the high tide at midnight on a full moon.

The problem is that there are 2 high tides a day about 12 hours apart. The only way to understand two high tides is to realize that there are two bulges of ocean water, one under the moon and one on the opposite side of the earth, as shown in Figure (16). In one 24 hour period we pass under both bulges.

Why is there a bulge on the backside? Why isn't the water all sucked up into one big bulge underneath the moon?

The answer is that the moon's gravity not only pulls on the earth's water, but on the earth itself. The force of gravity that the moon exerts on the earth is just the same

strength as the force the earth exerts on the moon. Since the earth is more massive, the effect on the earth is not as great, but it is noticeable. The reason for the second bulge of water on the far side of the earth is that the center of the earth is closer to the moon than the water on the back side, and therefore accelerates more rapidly toward the moon than the water on the back side. The water on the back side gets left behind to form a bulge.

The result, the fact that there are two high tides a day, the fact that there is a second bulge on the back side, is direct experimental evidence that the earth is accelerating toward the moon. It is direct evidence that the moon's gravity is pulling on the earth, just as the earth's gravity is pulling on the moon.

As a consequence of the earth's acceleration, the moon is not traveling in a circular orbit centered precisely on the center of the earth. Instead both the earth and the moon are traveling in circles about an axis point located on a line joining the earth's and moon's centers. This axis point is located much closer to the center of the earth than that of the moon, in fact it is located inside the earth about 3/4 of the way toward the earth's surface as shown in Figure (17).

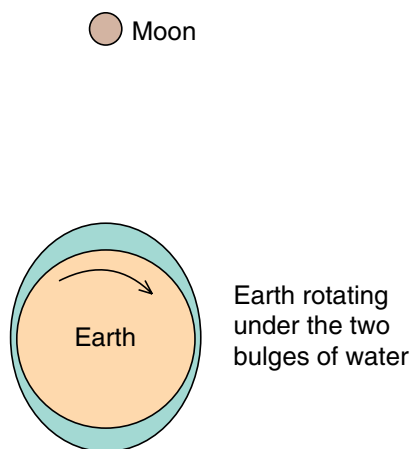


Figure 16
The two ocean bulges cause two high tides per day.

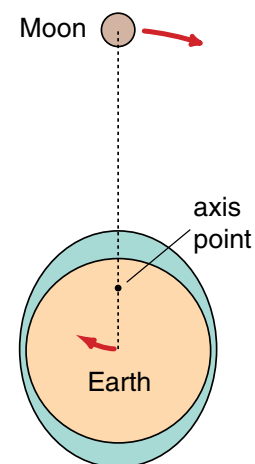


Figure 17
Both the earth and the moon travel in circular orbits about an axis point located about 1/4 of the way down below the earth's surface.

Planetary Units

In introductory physics texts, it has become almost an article of religion that all calculations shall be done using MKS units. This has some advantages – we do not have to talk about pounds and slugs, but practicing physicists seldom follow this rule. Physicists studying the behavior of elementary particles, for example, routinely use a system of units that simplify their calculations, units in which the speed of light and other fundamental constants have the numerical value 1. Using these special units they can quickly solve simple problems and gain an intuitive feeling for which quantities are important and which quantities are not.

In our work with projectiles in the lab the CGS system of units was excellent. The projectiles typically went distances from 10 to 100 cm, in times of the order of 1 second, and had masses of the order of 100 gm. There were no large exponents involved.

Now that we are studying the motion of earth satellites, we are faced with large exponents in quantities like the earth mass and the gravitational constant G which are 5.98×10^{24} kg and $6.67 \times 10^{-11} \text{ m}^3/\text{kg sec}^2$ respectively. The calculations we have done so far using these numbers have required a calculator, and we have had to work hard to gain insight from the results.

Table 1 Planetary Units

Constant	Symbol	Planetary units	MKS units
Gravitational Constant	G	20	$6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg sec}^2}$
Acceleration due to gravity at the earth's surface	g_e	20	9.8 m/sec ²
Earth mass	m_e	1	5.98×10^{24} kg
Moon mass	m_{moon}	.0123	7.36×10^{22} kg
Sun mass	m_{sun}	3.3×10^5	1.99×10^{30} kg
Metric ton	ton	1.67×10^{-22}	1000 kg
Earth radius	r_e	1	6.37×10^6 m
Moon radius	r_{moon}	.2725	1.74×10^6 m
Sun radius	r_{sun}	109	6.96×10^8 m
Earth orbit radius	$r_{\text{earth orbit}}$	23400	1.50×10^{11} m
Moon orbit radius	$r_{\text{moon orbit}}$	60	3.82×10^8 m
Hour	hr	1 hr	3600 sec
Moon period	lunar month (siderial)	656 hrs	2.36×10^6 sec (= 27.32 days)
Year	yr	8.78×10^3 hrs	3.16×10^7 sec

We now wish to introduce a new set of units, which we will call *planetary units*, that makes satellite calculations much simpler and more intuitive. One way to design a new set of units is to first decide what will be our unit mass, our unit length, and our unit time, and then work out all the conversion factors so that we can convert a problem into our new units. For working earth satellite problems, we have found that it is convenient to take the earth mass as the unit mass, the earth radius as the unit length, and the hour as the unit time.

$$m_{\text{earth}} = 1 \quad \text{earth mass}$$

$$R_{\text{earth}} = 1 \quad \text{earth radius}$$

$$\text{hour} = 1$$

With these choices, speed, for example, is measured in (earth radii)/hr, etc.

This system of units has a number of advantages. We can set m_e and r_e equal to 1 in the gravitational force formulas, greatly simplifying the results. We know immediately that a satellite has crashed if its orbital radius becomes less than 1. Typical satellite periods are a few hours and typical satellite speeds are from 1 to 10 earth radii per hour. What may be a bit surprising is that both the acceleration due to gravity at the surface of the earth, g , and Newton's universal gravitational constant G , have the same numerical value of 20.

Table 1 shows the conversion from MKS to planetary units of common quantities encountered in the study of satellites moving in the vicinity of the earth and the moon.

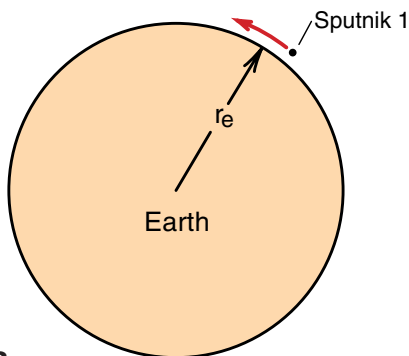


Figure 18
A satellite in a low earth orbit.

Exercise 6

We will have you convert Newton's universal gravitational constant G into planetary units. Start with

$$G = 6.67 \times 10^{-11} \frac{\text{meters}^3}{\text{kg sec}^2}$$

Then multiply or divide by the conversion factors

$$3600 \frac{\text{sec}}{\text{hr}}$$

$$5.98 \times 10^{24} \frac{\text{kg}}{\text{earth mass}}$$

$$6.37 \times 10^6 \frac{\text{meters}}{\text{earth radii}}$$

until all the dimensions in the formula for G are converted to planetary units. (I.e., convert from seconds to hours, kg to earth mass, and meters to earth radii.) If you do the conversion correctly, you should get the result

$$G = 20 \frac{(\text{earth radii})^3}{(\text{earth mass}) \text{hr}^2}$$

Exercise 7

Explain why g and G have the same numerical value in planetary units.

As an advertisement for how easy it is to use planetary units in satellite calculation, let us repeat Exercise (4) using these units. In that exercise we wished to calculate the period of Sputnik 1, a satellite traveling in a low earth orbit. We were to assume that Sputnik's orbital radius was essentially the earth's radius r_e as shown in Figure (18), and that Sputnik's acceleration toward the center of the earth was essentially the same as the projectiles we studied in the introductory lab, i.e., $g_e = 9.8 \text{ m/sec}^2$.

Using the formula

$$a = \frac{v^2}{r}$$

we get

$$g_e = \frac{v_{\text{Sputnik}}^2}{r_e} = \frac{v_{\text{Sputnik}}^2}{1}$$

Therefore

$$v_{\text{Sputnik}} = \sqrt{g_e} = \sqrt{20} \frac{\text{earth radii}}{\text{hr}}$$

Now the satellite travels a total distance $2\pi r_e$ to go one orbit, therefore the time it takes is

$$\text{Sputnik period} = \frac{2\pi r_e}{v_{\text{Sputnik}}} = \frac{2\pi}{\sqrt{20}} = 1.4 \text{ hrs}$$

Compare the algebra that we just did with what you had to go through to get an answer in Exercise (4). (You should have gotten the same answer, 1.4 hrs, or 84 minutes, or 5,040 seconds. This is in good agreement with the observed time for low orbit satellites.) If you have watched satellite launches on television, you may recall waiting about an hour and a half before the satellite returned.

Exercise 8

A satellite is placed in a circular orbit whose radius is $2r_e$ (it is one earth radii above the surface of the earth.)

- What is the acceleration due to gravity at this altitude?
- What is the period of this satellite's orbit?
- What is the shortest possible period any earth satellite can have? Explain your answer.

Exercise 9

Communication satellites are usually placed in circular orbits over the equator, at an altitude so that they take precisely 24 hours to orbit the earth. In this way they hover over the same point on the earth and can be in continuous communication with the same transmitters and receivers. This orbit is called the *Clarke orbit*, named after the science fiction writer Arthur Clark who first emphasized the importance of such an orbit. Calculate the radius of the Clark orbit.

COMPUTER PREDICTION OF SATELLITE ORBITS

In this chapter we have discussed two special kinds of motion that a projectile or satellite can have. One is the parabolic trajectory of a projectile thrown across the room – motion that is easily described by calculus and the constant acceleration formulas. The other is the orbital motion of the moon and man-made satellites that are in circular orbits. These orbits can be analyzed using the fact that their acceleration is directed toward the center of the circle and has a magnitude v^2/r .

These two examples are deceptively simple. Newton's diagram, Figure (11), shows that there is a continuous range of orbital shapes starting from simple projectile motion out to circular orbital motion and beyond. For all these orbital shapes, we know the projectile's acceleration is the gravitational acceleration toward the center of the earth. But to go from a knowledge of the acceleration to predicting the shape of the orbit is not necessarily an easy task.

There are no simple formulas like the constant acceleration formulas that allow us to predict where the satellite will be at any time in the future. Using advanced calculus techniques one can show that the orbits should have the shape of conic sections, one example being the elliptical orbits discovered by Kepler. But if we go to more complicated problems like trying to predict the motion of the Apollo 8 spacecraft from the earth to the moon and back, then a calculus approach is completely inadequate.

On the other hand these problems are easily handled using the step-by-step method of predicting motion, the method, discussed in Chapter 5, that we implement using the computer. With a slight modification of our old projectile motion program, we can predict what will happen to an earth satellite no matter how it is launched and what orbit it has. Adding a few more lines to the program allows us to send the satellite to the moon and back.

Once we are familiar with a basic satellite motion program, we can easily add new features. We can, for example, change the exponent in the gravitational force law from $1/r^2$ to $1/r^{2.1}$ to see what happens if the gravitational force law is modified. Similar modifica-

tions were in fact predicted by Einstein's general theory of relativity, thus we will be able to observe the kind of effects that were used to verify Einstein's theory.

New Computational Loop

In Chapter 5, we set up the machinery to do computer calculations. This involved learning the LET statement, constructing loops, plotting crosses, etc. Although this may have been a bit painful (but perhaps not as painful as learning calculus), we do not have to do much of that again. We can use essentially the same machinery to predict satellite orbits. The only significant change is in the calculational loop where we predict the particle's new position and velocity.

In the projectile motion program, the English version of the calculational loop was, from Figure (5-18)

```
! ----- Calculational Loop
DO
  LET  $\vec{R}_{\text{new}} = \vec{R}_{\text{old}} + \vec{V}_{\text{old}} * dt$ 
  LET  $\vec{A} = \vec{g}$ 
  LET  $\vec{V}_{\text{new}} = \vec{V}_{\text{old}} + \vec{A} * dt$ 
  LET  $T_{\text{new}} = T_{\text{old}} + dt$ 
  PLOT  $\vec{R}$ 
LOOP UNTIL T > 1
```

Figure 19

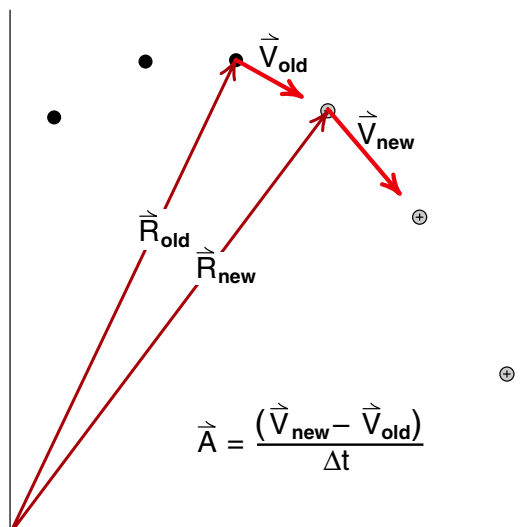


Figure 5-15a
Predicting the next new position.

This loop expresses the method of predicting motion that we developed from the analysis of strobe photographs. The idea behind the command

$$\text{LET } \vec{R}_{\text{new}} = \vec{R}_{\text{old}} + \vec{V}_{\text{old}} * dt$$

is illustrated in Figure (5-15a) reproduced here. The new position of the particle is obtained from the old position by adding the vector $\vec{V}_{\text{old}} * dt$ to the old coordinate vector \vec{R}_{old} .

Once we get to the new position of the particle, we need the new velocity vector in order to calculate the next new position. The new velocity vector is obtained from the command

$$\text{LET } \vec{V}_{\text{new}} = \vec{V}_{\text{old}} + \vec{A} * dt$$

as illustrated in Figure (5-15b). The DO-LOOP part of the program tells us to keep repeating this step-by-step process until we get as much of the trajectory as we want (in this case until one second has elapsed).

The calculational loop of Figure (19) works for projectile motion because we always know the projectile's acceleration \vec{A} which is given by the line

$$\boxed{\text{LET } \vec{A} = \vec{g}} \quad \text{projectile motion} \quad (16)$$

This is the line that characterizes projectile motion, the line that tells the computer that the projectile has a constant acceleration \vec{g} .

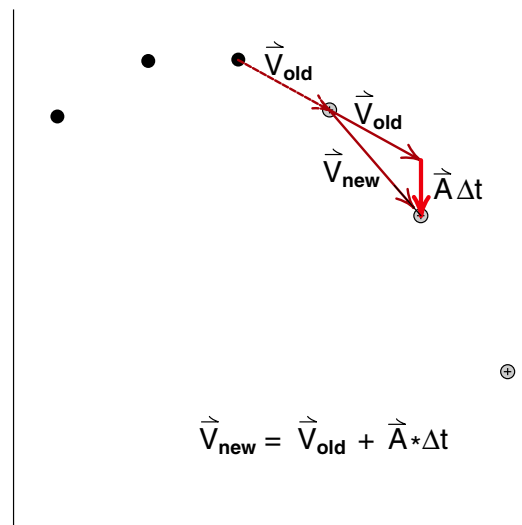


Figure 5-15b
Predicting the next new velocity.

The only fundamental change we need to make in going from projectile motion to satellite motion is to change our command for the particle's acceleration \vec{A} . Instead of assuming that the particle's acceleration is constant, we use Newton's law of gravity $|\vec{F}_g| = Gm_1m_2/r^2$ to calculate the force acting on the satellite, and then Newton's second law $\vec{A} = \vec{F}_g/m$ to obtain the resulting acceleration.

There are of course some other details. We have to find a way to express the vector nature of the gravitational force—i.e., to tell the computer which way the gravitational force is pointing, and we are going to change our plotting scale since we are no longer working in front of a 100 cm by 100 cm grid. But essentially we are replacing the command

$$\text{LET } \vec{A} = \vec{g}$$

by the new lines

$$\text{LET } \vec{F}_g = GM_em/R^2 \quad \text{with instructions for a direction}$$

$$\text{LET } \vec{A} = \vec{F}_g/m$$

and then using the same old program.

Unit Vectors

We have no problem describing the direction of the gravitational force on the satellite—the force is directed toward the center of the earth. But how do we tell the computer that? What mathematical technique can we use to express the direction of \vec{F}_g ?

The technique that we will use throughout the course is the use of the *unit vector*. **A unit vector is a dimensionless vector of length 1 that points in the direction of interest.** If we want a vector of length 5 newtons that points in the same direction, then we multiply our unit vector by the number 5 newtons to get the desired result. (Recall that multiplying a vector by a number, for example n , gives a vector n times as long, pointing in the same direction.)

There is an easy way to construct unit vectors. If we can find some vector that points in the desired direction, we divide that vector by its own length, and we end up with a vector of length 1, the required unit vector.

In our satellite motion problem, the gravitational force \vec{F}_g points toward the center of the earth. Thus to define the direction of \vec{F}_g , we need a unit vector that points toward the center of the earth. In Figure (20a) we show the coordinate vector \vec{R} which defines the position of the satellite in a coordinate system whose origin is at the center of the earth. In Figure (20b) we see the vector $-\vec{R}$, which points from the satellite to the center of the earth, the same direction as the gravitational force. Therefore we would like to turn $-\vec{R}$ into a unit vector, which we do by dividing by the length of R , namely the distance from the center of the earth to the satellite.

Since we will often use unit vectors in this text, we will designate them by a special symbol. Instead of an arrow over the letter, we will use what is called a *caret* by typographers, or more familiarly a *hat* by physicists. Thus our unit vector in the $-\vec{R}$ direction will be denoted by $-\hat{R}$ and is given by the formula

$$-\hat{R} = \frac{-\vec{R}}{R} \quad \text{unit vector in the } -\vec{R} \text{ direction} \quad (16)$$

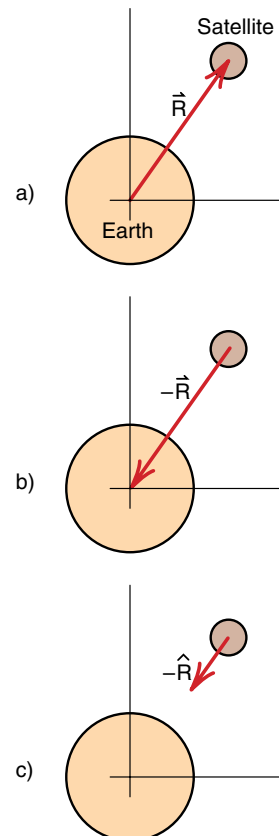


Figure 20
The unit vector $-\hat{R}$

In Equation (16), the length R is given by the Pythagorean theorem

$$R = \sqrt{R_x^2 + R_y^2} \quad (16a)$$

R_x and R_y being the x and y coordinates of the satellite.

With the unit vector $-\hat{R}$, we can now write an explicit formula for the gravitational force vector \vec{F}_g . We multiply the unit vector $-\hat{R}$ by the magnitude GMm/R^2 of the gravitational force to get

$$\vec{F}_g = \frac{GM_em}{R^2}(-\hat{R}) \quad (17)$$

Calculational Loop for Satellite Motion

We are now ready to go in an orderly way from the calculational loop for projectile motion to a calculational loop for satellite motion. We can focus our attention on the following three lines of the projectile motion calculation loop (Figure 21) because the other lines remain unchanged.

$$\begin{aligned} \text{LET } \vec{R}_{\text{new}} &= \vec{R}_{\text{old}} + \vec{V}_{\text{old}} * dt \\ \text{LET } \vec{A} &= \vec{g} \\ \text{LET } \vec{V}_{\text{new}} &= \vec{V}_{\text{old}} + \vec{A} * dt \end{aligned}$$

Figure 21

The first step is to replace $\text{LET } \vec{A} = \vec{g}$ by Newton's law of gravity and Newton's second law as shown in Figure (22).

$$\begin{aligned} \text{LET } \vec{R}_{\text{new}} &= \vec{R}_{\text{old}} + \vec{V}_{\text{old}} * dt \\ \text{LET } \vec{F}_g &= (-\hat{R})GM_em/R^2 \\ \text{LET } \vec{A} &= \vec{F}_g/m \\ \text{LET } \vec{V}_{\text{new}} &= \vec{V}_{\text{old}} + \vec{A} * dt \end{aligned}$$

Figure 22

Because BASIC is limited to working with numerical commands rather than vectors (an unfortunate limitation), the next step is to make sure that we can translate each of these vector commands into the separate x and y components. We will do this separately for each of the 4 lines.

The command

$$\text{LET } \vec{R}_{\text{new}} = \vec{R}_{\text{old}} + \vec{V}_{\text{old}} * dt$$

for \vec{R}_{new} becomes

$$\text{LET } R_x = R_x + V_x * dt \quad (18a)$$

$$\text{LET } R_y = R_y + V_y * dt \quad (18b)$$

where we drop the subscripts “new” and “old” because the computer automatically takes the old values on the right side of the LET statement, calculates a new value, and stores the new value in the memory cell named on the left side of the LET statement. (See our discussion of the LET statement on page 5-5).

In Equations (18a) and (18b) we obtain numerical values for the new coordinates R_x and R_y of the satellite. However, we will also need to know the distance R from the satellite to the center of the earth (in order to construct the unit vector $-\hat{R}$). The value of R is easily determined by adding the command

$$\text{LET } R = \text{SQR}(R_x * R_x + R_y * R_y) \quad (18c)$$

where SQR is BASIC's way of saying square root.

The translation of the command for \vec{F}_g only requires the translation of the unit vector \hat{R} into x and y coordinates. Remembering that $\hat{R} = \vec{R}/R$, we get

$$\hat{R}_x = R_x/R; \quad \hat{R}_y = R_y/R \quad (19)$$

thus the translation of the LET statement for \vec{F}_g can be written as

$$\text{LET } F_g = G * M_e * M / (R * R)$$

$$\text{LET } F_{gx} = (-R_x / R) * F_g$$

$$\text{LET } F_{gy} = (-R_y / R) * F_g$$

The computer can handle these lines because it already knows the new values of R_x , R_y and R from Equations (18a, b, and c).

The translation of LET statements for \vec{A} and \vec{V}_{new} are straightforward. We get

```
LET Ax = Fgx / M
LET Ay = Fgy / M

LET Vx = Vx + Ax * dt
LET Vy = Vy + Ay * dt
LET V = SQR(Vx*Vx + Vy*Vy)
```

We included a calculation of the magnitude V of the satellite's speed for future use. We may, for example, want to construct a unit vector in the $-\vec{V}$ direction to represent the direction of air resistance on a reentering satellite. We have found it convenient to routinely calculate the magnitude of any vector whose x and y coordinates we have just calculated.

Summary

To summarize our translation, we started with the vector commands

```
LET  $\vec{R}_{\text{new}}$  =  $\vec{R}_{\text{old}}$  +  $\vec{V}_{\text{old}}$  * dt
LET  $\vec{F}_g$  =  $(-\hat{R})GM_e m / R^2$ 
LET  $\vec{A}$  =  $\vec{F}_g / m$ 
LET  $\vec{V}_{\text{new}}$  =  $\vec{V}_{\text{old}}$  +  $\vec{A}$  * dt
```

and ended up with the BASIC commands

```
LET Rx = Rx + Vx * dt
LET Ry = Ry + Vy * dt
LET R = SQR (Rx*Rx + Ry*Ry)

LET Fg = G * Me * M / (R*R)
LET Fgx = (-Rx / R) * Fg
LET Fgy = (-Ry / R) * Fg

LET Ax = Fgx / M
LET Ay = Fgy / M

LET Vx = Vx + Ax * dt
LET Vy = Vy + Ay * dt
LET V = SQR(Vx*Vx + Vy*Vy)
```

Working Orbit Program

We are now ready to convert a working projectile motion program, Figure (5-23) reproduced here, into a working orbital motion program. In addition to converting the calculational loop as we have just discussed, we need to change some constants and plotting ranges, but the general structure of the program will be unchanged.

Plotting Window

We will initially consider satellite motion that stays reasonably close to the earth, within several earth radii. Using planetary units, and placing the earth at the center of the plot, we can get a reasonable range of orbits if we let R_x vary for example from -9 to $+9$ earth radii. If we have a standard 9" Macintosh screen, the x dimension should be 1.5 times the y dimension, thus R_y should go only from -6 to $+6$. The following command sets up this plotting window

```
SET WINDOW -9, 9, -6, 6
```

To show where the earth is located, we can use the following lines to plot a cross at the center of the earth

```
LET Rx = 0
LET Ry = 0
CALL CROSS
```

Constants and Initial Conditions

In going from the projectile motion to the satellite motion program, we have to change the constants and initial conditions. Using planetary units, our constants G , M_e , and m are

```
LET G = 20
LET Me = 1
LET m = .001
```

(Our choice of the satellite mass m does not matter because it cancels out of the calculation.)

For initial conditions, we will start the satellite .1 earth radii above the surface of the earth on the $+x$ axis;

```
LET Rx = 1.1
LET Ry = 0
LET R = SQR(Rx*Rx + Ry*Ry)
CALL CROSS
```

Projectile Motion Program

```

! ----- Plotting window
!           (x axis = 1.5 times y axis)
  SET WINDOW -40,140,-10,110

! ----- Draw & label axes
  BOX LINES 0,100,0,100
  PLOT TEXT, AT -3,0 : "0"
  PLOT TEXT, AT -13,96: "y=100"
  PLOT TEXT, AT 101,0 : "x=100"

! ----- Initial conditions
  LET DeltaT = .1
  LET Rx = 25.9
  LET Ry = 89.9
  LET Ux = (43.2 - 8.3)/(2*.1)
  LET Uy = (90.2 - 79.3)/(2*.1)
  LET T = 0
  CALL CROSS

! ----- Computer Time Step
  LET dt = .001
  LET i = 0

! ----- Computational loop
  DO
    LET Rx = Rx + Ux*dt
    LET Ry = Ry + Uy*dt
    LET Ax = 0
    LET Ay = -980
    LET Ux = Ux + Ax*dt
    LET Uy = Uy + Ay*dt
    LET T = T + dt
    LET i = i+1
    IF MOD(i,100) = 0 THEN CALL CROSS
    PLOT Rx,Ry
  LOOP UNTIL RX > 100

! ----- Subroutine "CROSS" draws
!           ! a cross at Rx,Ry.
  SUB CROSS
    PLOT LINES: Rx-2,Ry; Rx+2,Ry
    PLOT LINES: Rx,Ry-2; Rx,Ry+2
  END SUB
END

```

Figure 23

Projectile motion program that plots crosses every tenth of a second.

Orbit-1 Program

```

!----- Plotting window
!           (x axis = 1.5 times y axis))
  SET WINDOW -9,9,-6,6

!----- Plot cross at center of the Earth
  LET Rx = 0
  LET Ry = 0
  CALL CROSS

!----- Constants and Initial Conditions
  LET G = 19.91
  LET Me = 1
  LET M = .001

  LET Rx = 1.1
  LET Ry = 0
  LET R = SQR(Rx*Rx + Ry*Ry)
  CALL CROSS
  LET Ux = 0
  LET Uy = 5.5
  LET T = 0

!----- Time step
  LET dt = .01
  LET i = 0

!----- Caculational loop
  DO
    LET Rx = Rx + Ux*dt
    LET Ry = Ry + Uy*dt
    LET R = SQR(Rx*Rx + Ry*Ry)

    LET F = G*M*Me/R^2
    LET Fx = F*(-Rx/R)
    LET Fy = F*(-Ry/R)

    LET Ax = Fx/M
    LET Ay = Fy/M

    LET Ux = Ux + Ax*dt
    LET Uy = Uy + Ay*dt
    LET U = SQR(Ux*Ux + Uy*Uy)

    LET T = T + dt
    LET i = i + 1
    PLOT Rx,Ry
    IF MOD(i,40) = 0 THEN CALL CROSS
  LOOP UNTIL T > 8.45

!----- Subroutine "CROSS" draws
!           ! a cross at Rx,Ry.
  SUB CROSS
    PLOT LINES: Rx-.2,Ry; Rx+.2,Ry
    PLOT LINES: Rx,Ry-.2; Rx,Ry+.2
  END SUB
END

```

Figure 24

Our new orbital motion program.

We also calculated an initial value of R for use in the gravitational force formula, and plotted a cross at this initial point.

We are going to fire the satellite in the $+y$ direction, parallel to the surface of the earth. Trial and error shows us that a reasonable value for the speed of the satellite is 5.5 earth radii per hour, thus we write for our initial velocity commands

```
LET Vx = 0
LET Vy = 5.5
LET V = SQR(Vx*Vx + Vy*Vy)
```

In our projectile motion program of Figure (5-23) we wanted a cross plotted every 100 time steps dt . This was done with the command

```
IF MOD(i,100) = 0 THEN CALL CROSS
```

For our orbit program, trial and error shows that we get a good looking plot if we draw a cross every 40 time steps, each time step dt being .01 hours. Thus our new MOD line will be

```
IF MOD(i,40) = 0 THEN CALL CROSS
```

and we will get a cross every $.01 * 40 = .4$ hours.

The final change is to stop plotting after one orbit. From running the program we find that one orbit takes about 9 hours, thus we can stop plotting just before one orbit with the LOOP instruction

```
LOOP UNTIL T > 9
```

Putting all these steps together gives us the complete BASIC program shown in Figure (24).

When we run the Orbit 1 program, we get the elliptical orbit shown in Figure (25).

Exercise 10

Convert your projectile motion program to the Orbit 1 program. Use the same initial conditions so that you get the same orbit as that shown in Figure (25). (It is important to get your Orbit 1 program running correctly now, for it will be used as the basis for studying several phenomena during the rest of this chapter. If you are having problems, simply type the program in precisely as shown in Figure (24).)

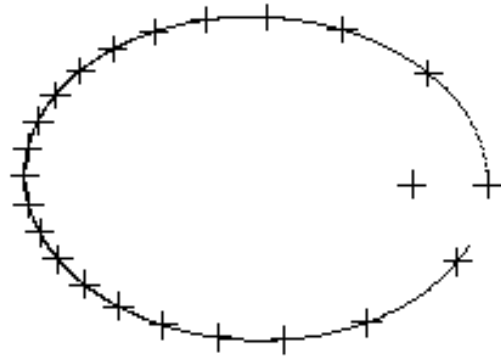


Figure 25
Output of the Orbit 1 program. The satellite is initially out at a distance $x = 1.1$ earth radii, and is fired in the $+y$ direction at a speed of 5.5 earth radii per hour.

Once your program is working, it is easy to make small modifications to improve the results. To create Figure (25a) we added the command

```
BOX CIRCLE -1,1,-1,1
```

to draw a circle to represent the earth. We also changed dt to .001 and changed the MOD command to MOD(i,539) to get an even number of crosses around the orbit. We then plotted until $T = 9$ hours. (With dt ten times smaller, our i counter has to be ten times bigger to get the old crosses.)

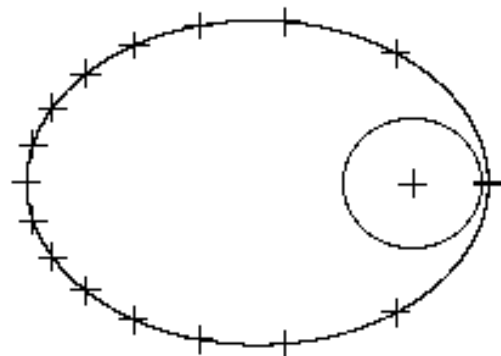


Figure 25a

Satellite Motion Laboratory

In our study of projectile motion, we could go to the laboratory and take strobe photographs in order to see how projectiles behaved. Obtaining experimental data for the study of satellite motion is somewhat more difficult. What we will do is to use the Orbit 1 program or slight modification of it to stimulate satellite motion, using it as our laboratory for the study of the behavior of satellites.

But first we wish to check that the Orbit 1 program makes predictions that are in agreement with experiment. The program is based on Newton's laws of gravity, $F_g = GMm/r^2$, Newton's law of motion $\vec{a} = \vec{F}/m$, and the procedures we developed earlier for predicting the motion of an object whose acceleration is known. Thus a verification of the results of the Orbit 1 program can be considered a verification of these laws and procedures.

Some tests of the Orbit 1 program can be made using the results of your own experience. Anyone who has listened to the launch of a low orbit satellite should be aware that the satellite takes about 90 minutes to go around the earth once. The Orbit 1 program should give the same result, which you can check in Exercise 11. Another obvious test is the prediction of the period of the moon in its orbit around the earth. It is about 4 weeks from full moon to full moon, thus the period should be approximately 4 weeks or 28 days. The fact that the apparent diameter of the moon does not change much during this time indicates that the moon is traveling in a nearly circular orbit about the earth. If you accept the astronomer's measurements that the moon orbit radius is about 60 earth radii away, then you can check the Orbit 1 program to see if it predicts a 4 week period for an earth satellite in a circular orbit of that radius (Exercise 12).

(An easy way to measure the distance to the moon was provided by the first moon landing. Because of a problem with Neil Armstrong's helmet, radio signals sent to Neil from Houston were retransmitted by Neil's microphone, giving an apparent echo. The echo was particularly noticeable while Neil was setting up a TV camera. On a tape of the mission supplied by NASA,

you can hear the statement "That's good there, Neil". A short while later you hear the clear echo "That's good there, Neil". The time delay from the original statement and the echo is the time it takes a radio wave, traveling at the speed of light, to go to the moon and back. Using an inexpensive stop watch, one can easily measure the time delay as being about $2\frac{2}{5}$ seconds. Thus the one-way trip to the moon is $1\frac{1}{5}$ seconds. Since light travels 1 ft/nanosecond, or 1 billion feet per second, from this one determines that the moon is about 1.2 billion feet away. You can convert this distance to earth radii to check the astronomer's value of 60 earth radii as the average distance to the moon.)

Exercise 11

Adjust the initial conditions in your Orbit 1 program so that the satellite is in a low earth orbit, and see what the period of the orbit is.

(To adjust the initial conditions, start, for example, with $R_x = 1.01$, $R_y = 0$, $v_x = 0$ and adjust v_y until you get a circular orbit centered on the earth. As a check that the satellite did not go below the surface of the earth, you could add the line

```
IF R < 1 THEN PRINT "CRASHED"
```

Adding this line just after you have calculated R in the DO LOOP will immediately warn you if the satellite has crashed. You can then adjust the initial v_y so that you just avoid a crash. Once you have a circular orbit, you can adjust the time in the "LOOP UNTIL T > ..." command so that just one orbit is printed. This tells you how long the orbit took. You can also see how long the orbit took by adding the line in the DO LOOP

```
IF MOD(I, 40) = 0 THEN PRINT T, RX, RY
```

Looking at the values of R_x and R_y you can tell when one orbit is completed, and the value of T tells you how long it took.

Exercise 12

Put the satellite in a circular orbit whose radius is equal to the radius of the moon's orbit. (See Table 1, Planetary Units, for the value of the moon orbit radius.) See if you predict that the moon will take about 4 weeks to go around this orbit.)

KEPLER'S LAWS

A more detailed test of Newton's laws and the Orbit 1 program is provided by Kepler's laws of planetary motion.

To get a feeling for the problems involved in studying planetary motion, imagine that you were given the job of going outside, looking at the sky, and figuring out how celestial objects moved. The easiest to start with is the moon, which becomes full again every four weeks. On closer observation you would notice that the moon moved past the background of the apparently fixed stars, returning to its original position in the sky every 27.3 days. Since, as we mentioned, the diameter of the moon does not change much, you might then conclude that the moon is in a circular orbit about the earth, with a period of 27.3 days.

The time it takes the moon to return to the same point in the sky is not precisely equal to the time between full moons. A full moon occurs when the sun, earth, and moon are in alignment. If the sun itself appears to move relative to the fixed stars, the full moons will not occur at precisely the same point, and the time between full moons will not be exactly the time it takes the moon to go around once.

To study the motion of the sun past the background of the fixed stars is more difficult because the stars are not visible when the sun is up. One way to locate the position of the sun is to observe what stars are overhead at "true" midnight, half way between dusk and dawn. The sun should then be located on the opposite side of the sky. (You also have to correct for the north/south position of the sun.) After a fair amount of observation and calculations, you would find that the sun itself moves past the background of the fixed stars, returning to its starting point once a year.

From the fact that the sun takes one year to go around the sky, and the fact that its apparent diameter remains essentially constant, you might well conclude that the sun, like the moon, is traveling in a circular orbit about the earth. This was the accepted conclusion by most astronomers up to the time of Nicolaus Copernicus in the early 1500s AD.

If you start looking at the motion of the planets like Mercury, Venus, Mars, Jupiter, and Saturn, all easily

visible without a telescope, the situation is more complicated. Mars, for example, moves in one direction against the background of the fixed stars, then reverses and goes backward for a while, then forward again as shown in Figure (26). None of the planets has the simple uniform motion seen in the case of the moon and the sun.

After a lot of observation and the construction of many plots, you might make a rather significant discovery. You might find what the early Greek astronomers learned, namely that if you assume that the planets Mercury, Venus, Mars, Jupiter, and Saturn travel in circular orbits about the sun, while the sun is traveling in a circular orbit about the earth, then you can explain all the peculiar motion of the planets. This is a remarkable simplification and compelling evidence that there is a simple order underlying the motion of celestial objects.

One of the features of astronomical observations is that they become more accurate as time passes. If you observe the moon for 100 orbits, you can determine the average period of the moon nearly 100 times more accurately than from the observation of a single period. You can also detect any gradual shift of the orbit 100 times more accurately.

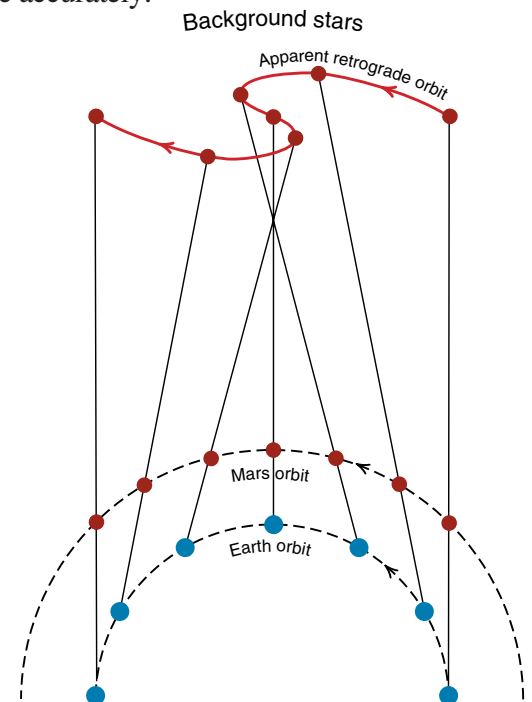


Figure 26
Retrograde motion of the planet Mars. Modern view of why Mars appears to reverse its direction of motion for a while.

Even by the time of the famous Greek astronomer Ptolemy in the second century AD, observations of the positions of the planets had been made for a sufficiently long time that it had become clear that the planets did not travel in precisely circular orbits about the sun. Some way was needed to explain the non circularity of the orbits.

The simplicity of a circular orbit was such a compelling idea that it was not abandoned. Recall that the apparently peculiar motion of Mars could be explained by assuming that Mars traveled in a circular orbit about the sun which in turn traveled in a circular orbit about the earth. By having circular orbits centered on points that are themselves in circular orbits, you can construct complex orbits. By choosing enough circles with the correct radii and periods, you can construct any kind of orbit you wish.

Ptolemy explained the slight variations in the planetary orbits by assuming that the planets traveled in circles around points which traveled in circles about the sun, which in turn traveled in a circle about the earth. The extra cycle in this scheme was called an *epicycle*. With just a few epicycles, Ptolemy was able to accurately explain all observations of planetary motion made by the second century AD.

With 1500 more years of planetary observations, Ptolemy's scheme was no longer working well. With far more accurate observations over this long span of time, it was necessary to introduce many more epicycles into Ptolemy's scheme in order to explain the positions of the planets.

Even before problems with Ptolemy's scheme became apparent, there were those who argued that the scheme would be simpler if the sun were at the center of the solar system and all the planets, including the earth, moved in circles about the sun. This view was not taken seriously in ancient times, because such a scheme would predict that the earth was moving at a tremendous speed, a motion that surely would be felt. (The principle of relativity was not understood at that time.)

For similar reasons, one did not use the rotation of the earth to explain the daily motion of sun, moon, and stars. That would imply that the surface of the earth at the equator would be moving at a speed of around a thousand miles per hour, an unimaginable speed!

In 1543, Nicolaus Copernicus put forth a detailed plan for the motion of the planets from the point of view that the sun was the center of the solar system and that all the

planets moved in circular orbits about the sun. Such a theory not only conflicted with common sense about feeling the motion of the earth, but also displaced the earth and mankind from the center of the universe, two results quite unacceptable to many scholars and theologians.

Copernicus' theory was not quite as simple as it first sounds. Because of the accuracy with which planetary motion was known by 1543, it was necessary to include epicycles in the planetary orbits in Copernicus' model.

Starting around 1576, the Dutch astronomer Tycho Brahe made a series of observations of the planetary positions that were a significant improvement over previous measurements. This work was done before the invention of the telescope, using apparatus like that shown in Figure (27). Tycho Brahe did not happen to believe in the Copernican sun-centered theory, but that had little

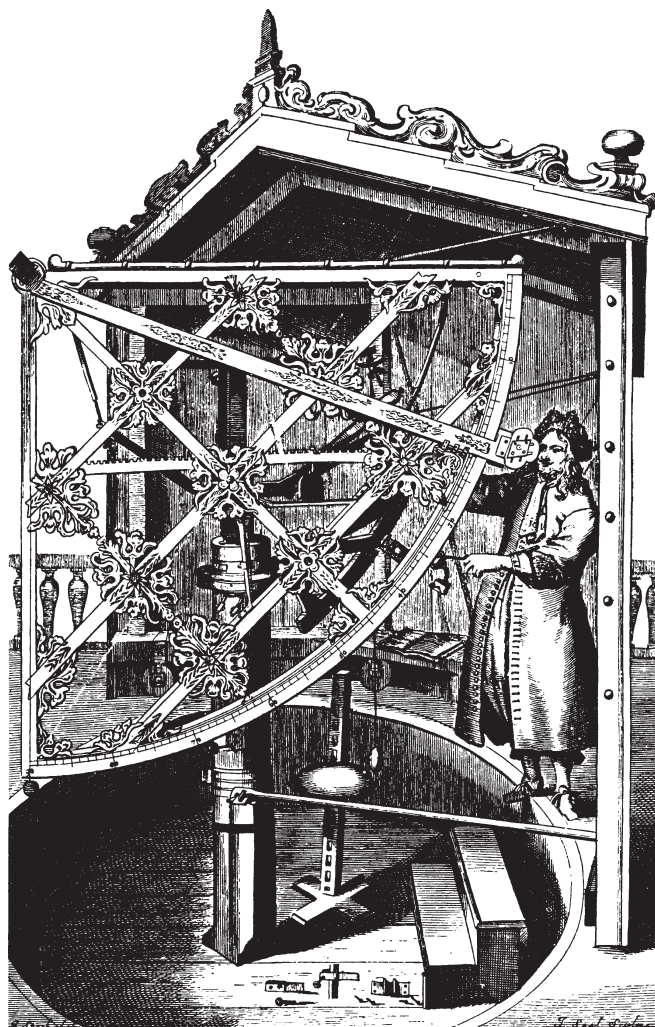


Figure 27
Tycho Brahe's apparatus.

effect on the reason for making the more accurate observations. Both the Ptolemaic and Copernican systems relied on epicycles, and more accurate data was needed to improve the predictive power of these theories.

Johannes Kepler, a student of Tycho Brahe, started from the simplicity inherent in the Copernican system, but went one step farther than Copernicus. Abandoning the idea that planetary motion had to be described in terms of circular orbits and epicycles, Kepler used Tycho Brahe's accurate data to look for a better way to describe the planet's motion. Kepler found that the planetary orbits were accurately and simply described by ellipses, where the sun was at one of the foci of the ellipse. (We will soon discuss the properties of ellipses.) Kepler also found a simple rule relating the speed of the planet to the area swept out by a line drawn from the planet to the sun. And thirdly, he discovered that the ratio of the cube of the orbital radius to the square of the period was the same for all planets. These three results are known as Kepler's three laws of planetary motion.

Kepler's three simple rules for planetary motion, which we will discuss in more detail shortly, replaced and improved upon the complex system of epicycles needed by all previous theories. After Kepler's discovery, it was obvious that the sun-centered system and elliptical orbits provided by far the simplest description of the motion of the heavenly objects. For Isaac Newton, half a century later, Kepler's laws served as a fundamental test of his theories of motion and gravitation. We will now use Kepler's laws in a similar way, as a test of the validity of the Orbit 1 program and our techniques for predicting motion.

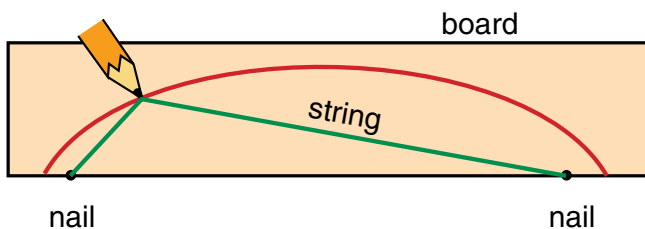


Figure 28
Ellipse constructed with two nails and a string.

Kepler's First Law

Kepler's first law states that the planets move in elliptical orbits with the sun at one focus. By analogy we should find from our Orbit 1 program that earth satellites move in elliptical orbits with the center of the earth at one focus. To check this prediction, we need to know how to construct an ellipse and determine where the focus is located.

The arch above the entrance to many of the old New England horse sheds was a section of an ellipse. The carpenters drew the curve by placing two nails on a wide board, attaching the ends of a string to each nail, and moving a pencil around while keeping the string taut as shown in Figure (28). The result is half an ellipse with a nail at each one of the foci. (If you are in the Mormon Tabernacle's elliptical auditorium and drop a pin at one focus, the pin drop can be heard at the other focus because the sound waves bouncing off the walls all travel the same distance and add up constructively at the second focus point.)

To see if the satellite orbit from the Orbit 1 program is an ellipse, we first locate the second focus using the output shown in Figure (25a) by locating the point symmetrically across from the center of the earth as shown in Figure (29). Then at several points along the orbit we draw lines from that point to each focus as shown, and see if the total length of the lines (what would be the length of the stretched string) remains constant as we go around the orbit).

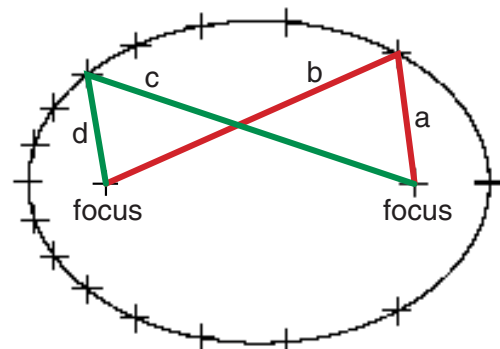


Figure 29
Checking that our satellite orbits are an ellipse. We construct a second focus, and then see if the sum of the distances from each focus to a point on the ellipse is the same for any point around the ellipse. For this diagram, we should show that $a+b = c+d$.

Exercise 13

Using the output from your Orbit 1 program, check that the orbit is an ellipse.

Exercise 14

Slightly alter the initial conditions of your Orbit 1 program to get a different shaped orbit. (Preferably, make the orbit more stretched out.) Check that the resulting orbit is still an ellipse.

Kepler's Second Law

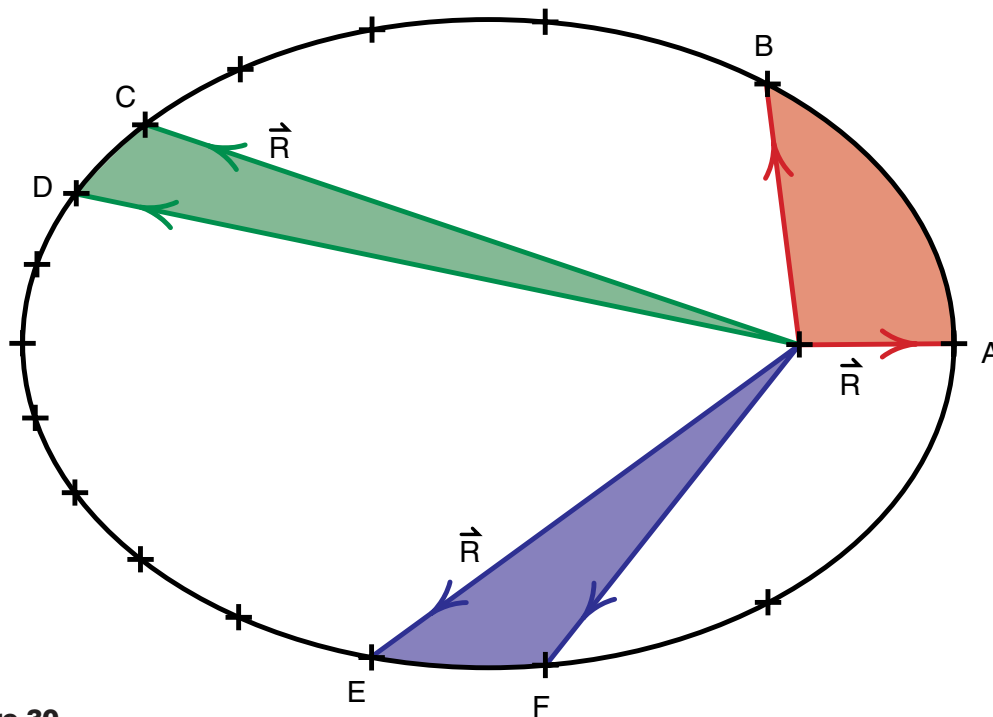
Kepler's second law relates the speed of the planet to the area swept out by a line connecting the sun to the planet. If we think of the sun as being at the origin of the coordinate system, then the line from the sun to the planet is what we have been calling the *coordinate vector* \vec{R} . It is also called the *radius vector* \vec{R} . Kepler's second law explicitly states *that the radius vector \vec{R} sweeps out equal areas in equal times*.

To apply Kepler's second law to the output of our Orbit 1 program, we note that we had the computer plot a cross at equal times along the orbit. Thus the area swept out by the radius vector should be the same as \vec{R} moves from one cross to the next. To check this prediction, we

have in Figure (30) reproduced the output of Figure (25a), shaded the areas swept out as \vec{R} moves from positions A to B, from C to D, and from E to F. These areas should look approximately equal; you will check that they are in fact equal in Exercise 15.

The most significant consequence of Kepler's second law is that in order to sweep out equal areas while the radius vector is changing length, the planet or satellite must move more rapidly when the radius vector is short, and more slowly when the radius vector is long. The planet moves more rapidly when in close to the sun, and more slowly when far away.

An extreme example of elliptical satellite orbits are the orbits of some of the comets that periodically visit the sun. Halley's comet, for example, visits the sun once every 76 years. The comet spends about 1 year in the close vicinity of the sun, where it is visible from the earth, and the other 75 years on the rest of its orbit which goes out beyond the edge of the planetary system. The comet moves rapidly past the sun, and spends the majority of the 76 year orbital period creeping around the back side of its orbit where its radius vector is very long.

**Figure 30**

Kepler's Second Law. The radius vector \vec{R} should sweep out equal areas in equal time.

Exercise 15

For both of your plots from Exercises 13 and 14, check that the satellite's radius vector sweeps out equal areas in equal times. Explicitly compare the area swept out during a time interval where the satellite is in close to the earth to an equal time interval where the satellite is far from the earth.

This exercise requires that you measure the areas of lopsided pie-shaped sections. There are a number of ways of doing this. You can, for example, draw the sections out on graph paper and count the squares, you can break the areas up into triangles and calculate the areas of the triangles, or you can cut the areas out of cardboard and weigh them.

Kepler's Third Law

Kepler's third law states that the ratio of the cube of the orbital radius R to the square of the period T is the same ratio for all the planets. We can easily use Newton's laws of gravity and motion to check this result for the case of circular orbits. The result, which you are to calculate in Exercise 16, is

$$\frac{R^3}{T^2} = \frac{GM_s}{4\pi^2} \quad (20)$$

where M_s is the mass of the sun. In this calculation, the mass m_p of the planet, the orbital radius R , the speed v all cancelled, leaving only the sun mass M_s as a variable. Since all the planets orbit the same sun, this ratio should be the same for all the planets.

When the planet is in an elliptical orbit, the length of the radius vector \hat{R} changes as the planet goes around the sun. What Kepler found was that the ratio of R^3/T^2 was constant if you used the "semi major axis" for R . The semi major axis is the half the maximum diameter of the ellipse, shown in Figure (31). As an optional Exercise (17), you can compare the ratio of R^3/T^2 for the two elliptical orbits of Exercises (13) and (14), using the semi major axis for R .

Exercise 16

Consider the example of a planet of mass m_p in a circular orbit about the sun whose mass is M_s . Using Newton's second law and Newton's law of gravity, and the fact that for circular motion the magnitude of the acceleration is v^2/R , solve for the radius R of the orbit. Then use the fact that the period T is the distance $2\pi R$ divided by the speed v , and construct the ratio R^3/T^2 . All the variables except M_s should cancel and you should get the result shown in Equation 20.

Exercise 17 (optional)

A more general statement of Kepler's third law, that applies to elliptical orbits, is that R^3/T^2 is the same for all the planets, where R is the semi major axis of the ellipse (as shown in Figure (31)). Check this prediction for the two elliptical orbits used in Exercises (13) and (14). In both of those examples the satellite was orbiting the same earth, thus the ratios should be the same.

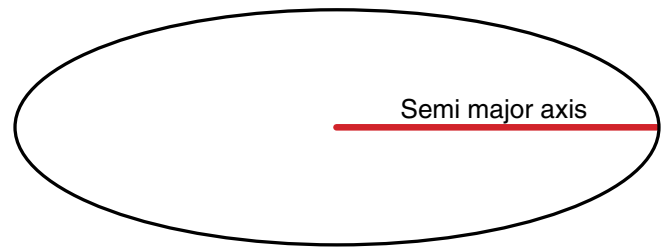


Figure 31
The semi major axis of an ellipse.

MODIFIED GRAVITY AND GENERAL RELATIVITY

After we have verified that the Orbit 1 program calculates orbits that are in agreement with Kepler's laws of motion, we should be reasonably confident that the program is ready to serve as a laboratory for the study of new phenomena we have not necessarily encountered before. To illustrate what we can do, we will begin with a question that cannot be answered in the lab. What would happen if we modified the law of gravity? What, for example, would happen if we changed the universal constant G , or altered the exponent on the r dependence of the force? With the computer program, these questions are easily answered. We simply make the change and see what happens.

These changes should not be made completely without thought. I have seen a project where a student tried to observe the effect of changing the mass of the satellite. After many plots, he concluded that the effect was not great. That is not a surprising result considering the fact that the mass m_s of the satellite cancels out when you equate the gravitational force to $m_s \vec{a}$.

One can also see that, as far as its effect on a satellite's orbit, changing the universal constant G will have an effect equivalent to changing the earth mass M_e . Since Kepler's laws did not depend particularly on what mass our sun had, one suspects that Kepler's laws should also hold when G or M_e are modified. This guess can easily be checked using the Orbit 1 program.

Changing the r dependence of the gravitational force is another matter. After developing the special theory of relativity, Einstein took a look at Newton's theory of gravity and saw that it was not consistent with the principle of relativity. For one thing, because the Newtonian gravitational force is supposed to point to the current instantaneous position of a mass, it should be possible using Newtonian gravity to send signals faster than the speed of light. (Think about how you might do that.)

From the period of time between 1905 and 1915 Einstein worked out a new theory of gravity that was consistent with special relativity and, in the limit of slowly moving, not too massive objects, gave the same results as Newtonian gravity. We will get to see how this process works when, in the latter half of this text we

start with Coulomb's electric force law, include the effects of special relativity, and find that magnetism is one of the essential consequences of this combination.

Einstein's relativistic theory of gravity is more complex than the theory of electricity and magnetism, and the new predictions of the theory are much harder to test. It turns out that Newtonian gravity accurately describes almost all planetary motion we can observe in our solar system. Einstein calculated that his new theory of gravity should predict new observable effects only in the case of the orbit of Mercury and in the deflection of starlight as it passed the rim of the sun. In 1917 Sir Arthur Eddington led a famous eclipse expedition in which the deflection of starlight past the rim of the eclipsed sun could be observed. The deflection predicted by Einstein was observed, making this the first clear correction to Newtonian gravity detected in 250 years. Einstein's real fame began with the success of the Eddington expedition.

While Einstein set out to construct a theory of gravity consistent with special relativity, he was also impressed by the connection between gravity and space. Because all projectiles here on the surface of the earth have the same downward acceleration, if you were in a sealed room you could not be completely sure whether your room was on the surface of the earth, and the downward accelerations were caused by gravity, or whether you were out in space, and your room was accelerating upward with an acceleration g . These equivalent situations are shown in Figure (32).

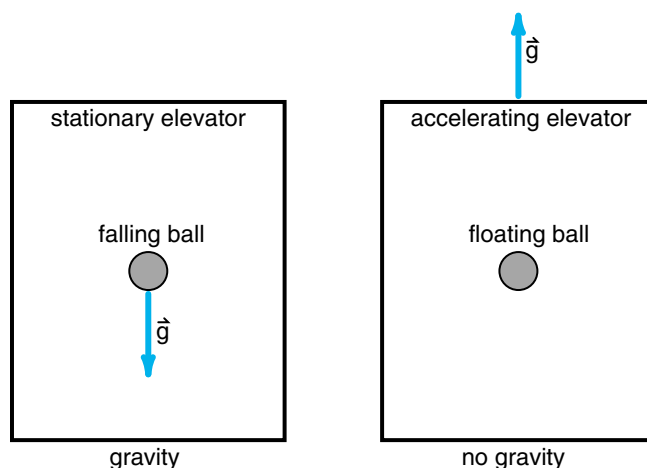


Figure 32
Equivalent situations. Explain why you would feel the same forces if you were sitting on the floor of each of the two rooms.

The equivalence between a gravitational force and an acceleration turned out to be the cornerstone of Einstein's relativistic theory of gravity. It turned out that Einstein's new theory of gravity could be interpreted as a theory of space and time, where mass caused a curvature of space, and what we call gravitational forces were a consequence of this curvature of space. This geometrical theory of gravity, Einstein's relativistic theory, is commonly called the *General Theory of Relativity*.

As they often say in textbooks, a full discussion of Einstein's relativistic theory of gravity is "beyond the scope of this text". However we can look at at least one of the predictions. As far as satellite orbit calculations are concerned, we can think of Einstein's theory as a slight modification of the Newtonian theory. We have seen that any modification of the factors G , m_s or m_e in the Newtonian gravitational force law would not have a detectable effect. The only thing we could notice is some change in the exponent of r .

With a few of quick runs of the Orbit 1 program, you will discover that the satellite orbit is very sensitive to the exponent of r . In Figure (33) we have changed the exponent from -2 to -1.9 . This simply requires changing

$$G * m_s * m_e / (R \wedge 2)$$

to

$$G * m_s * m_e / (R \wedge 1.9)$$

in the formula for F_g . The result is a striking change in the orbit. When the exponent is -2 , the elliptical orbit is rock steady. When we change the exponent to -1.9 , the ellipse starts rotating around the earth. This rotation of the ellipse is called the *precession of the perihelion*, where the word "perihelion" describes the line connecting the two foci of the ellipse.

A $1/r^2$ force law is unique in that only for this exponent, -2 , does the perihelion, the axis of the elliptical orbit, remain steady. For any other value of the exponent, the perihelion rotates or precesses one way or another.

It turns out that a number of effects can cause the perihelion of a planet's orbit to precess. The biggest effect we have not yet discussed is the fact that there are a number of planets all orbiting the sun at the same time, and these planets all exert slight forces on each other. These slight forces cause slight perihelion precessions.

In the 250 years from the time of Newton's discovery of the law of gravity, to the early 1900s, astronomers carefully worked out the predicted orbits of the planets, including the effects of the forces between the planets themselves. This work, done before the development of computers, was an extremely laborious task. A good fraction of one's lifetime work could be spent on a single calculation.

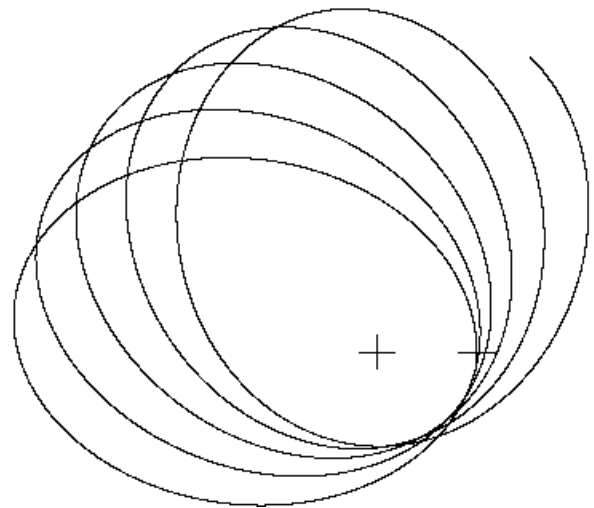


Figure 33
Planetary orbit when the gravitational force is modified to a $1/r^{1.9}$ force.

The orbit of the planet Mercury provided a good test of these calculations because its orbital ellipse is more extended than that of the other close-in planets. The more extended an ellipse, the easier it is to observe a precession. (You cannot even detect a precession for a circular orbit.) Mercury's orbit has a small but observable precession. Its orbit precesses by an angle that is slightly less than .2 degrees every *century*. This is a very small precession which you could never detect in one orbit. But the orbit of Mercury has been observed for about 3000 years, or 30 centuries. That is over a 5 degree precession which is easily detectable.

When measuring small angles, astronomers divide the degree into 60 *minutes of arc*, and for even smaller angles, divide the minute into 60 *seconds of arc*. One second of arc, 1/3600 of a degree, is a very small angle. A basketball 30 miles distant subtends an angle of about 1 second of arc. In these units, Mercury's orbit precesses about 650 seconds of arc per century.

By 1900, astronomers doing Newtonian mechanics calculations could account for all but 43 seconds of arc per century precession of Mercury's orbit as being caused by the influence of neighboring planets. The 43 seconds of arc discrepancy could not be explained. One of the important predictions of Einstein's relativistic theory of gravity is that it predicts a 43 second of arc per century precession of Mercury's orbit, a precession caused by a change in the gravitational force law and not due to neighboring planets. Einstein used this explanation of the 43 seconds of arc discrepancy as the main experimental foundation for his relativistic theory of gravity when he just presented it in 1915. The importance of the Eddington eclipse expedition in 1917 is that a completely new phenomena, predicted by Einstein's theory, was detected.

(The Eddington expedition verified more than just the fact that light is deflected by the gravitational attraction of a star. You can easily construct a theory where the energy in the light beam is related to mass via the formula $E = mc^2$, and then use Newtonian gravity to predict a deflection. Einstein's General Relativity predicts a deflection twice as large as this modified Newtonian approach. The Eddington expedition observed the larger prediction of General Relativity, providing convincing evidence that General Relativity rather than Newtonian gravity was the more correct theory of gravity.)

Exercise 18

Start with your Orbit 1 program, modify the exponent in the gravitational force law, and see what happens. Begin with a small modification so that you can see how to plot the results. (If you make a larger modification, you will have to change the plotting window to get interesting results.)

(To get the 43 seconds of arc per century precession of Mercury's orbit, using a modified gravitational force law, the force should be proportional to $1/r^{2.00000016}$ instead of $1/r^2$.)

CONSERVATION OF ANGULAR MOMENTUM

With the ability to work with realistic satellite orbits rather than just the circular orbits, we will be able to make significant tests of the laws of conservation of angular momentum and of energy, as applied to satellite motion. In this section, we will first see how Kepler's second law of planetary motion is a direct consequence of the conservation of angular momentum, and then do some calculations with the Orbit 1 program to see that a satellite's angular momentum is in fact conserved—does not change as the satellite goes around the earth. In the next section we will first take a more general look at the idea of a conservation law, and then apply this discussion to the conservation of energy for satellite orbits.

Recall that Kepler's second law of planetary motion states that a line from the sun to the planet, the radius vector, sweeps out equal areas in equal times. For this to be true when the planet is in an elliptical orbit, the planet must move faster when in close to the sun and the radius vector is short, and slower when far away and the radius vector is long.

To intuitively see that this speeding up and slowing down is a consequence of the conservation of angular momentum, one can modify the three dumbbell experiment we used to demonstrate the conservation of

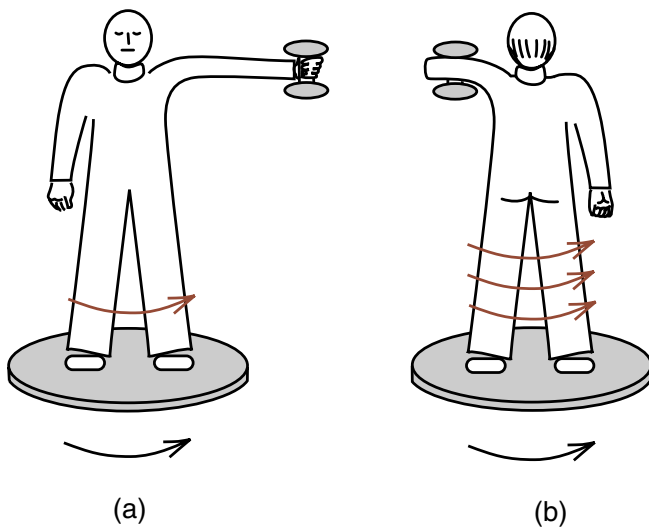


Figure 34
One dumbbell experiment.

angular momentum. In this demonstration the instructor uses only one dumbbell. After a student assists the instructor in getting his rotation started, the instructor extends the dumbbell out to full arm's reach, for instance, when he is facing the class, and pulls his arm in when he is facing away as shown in Figure (34). Some practice is needed to maintain this pattern and not lose one's balance.

The rather expected result of this demonstration is that the instructor rotates more slowly when his arm is far out, and more rapidly when his arm is in close. If we associate the dumbbell with a satellite orbiting the earth, we see the same speeding up as the lever arm about the axis of rotation is reduced, and slowing down as the lever arm is increased.

A fairly simple geometrical construction demonstrates that the rule about the radius vector sweeping out equal areas in equal times is precisely what is required for conservation of angular momentum. In Figure (35a) we have plotted an elliptical satellite orbit showing the position of the planet for two different equal time intervals. The time intervals Δt are short enough that we can fairly accurately represent the displacement of the satellite by short, straight lines of length $v_1 \Delta t$ in the upper triangle and $v_2 \Delta t$ in the lower triangle. With this approximation we can represent the areas swept out by the radius vector by triangles as shown by the shaded areas in Figure (35a).

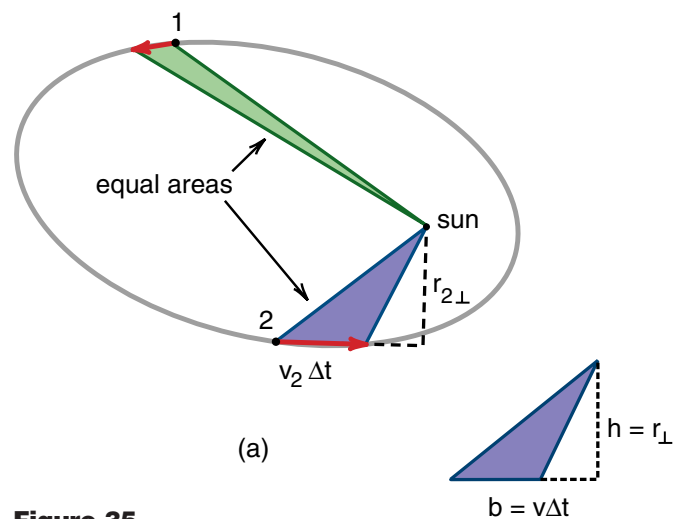


Figure 35
Calculating the area swept out by the planet during a short time interval Δt .

Now the area of a triangle is one half the base times the altitude. If you look at the lower triangle in Figure (35a), and take the side $v_2\Delta t$ as the base, then the distance labeled $r_{2\perp}$ is the altitude, as seen in the sketch in Figure (34b). Thus the area of the triangle at position 2 is

$$\left. \begin{array}{l} \text{area swept} \\ \text{out at position 2} \\ \text{in a time } \Delta t \end{array} \right\} = \frac{1}{2} (\text{base}) \times (\text{altitude}) \quad (21)$$

$$= \frac{1}{2} (v_2\Delta t) \times r_{2\perp}$$

When the satellite is at position 2 in Figure (35a), moving at a velocity \vec{v}_2 , the distance of closest approach if it continued at the same velocity \vec{v}_2 would be the distance $r_{2\perp}$. Thus $r_{2\perp}$ is the “lever arm” for the motion of the satellite at this point in the orbit.

We get a similar formula for the area of the triangle at position 1. Using Kepler’s second law which says that these areas should be equal for equal times Δt , we get

$$\frac{1}{2}(v_1\Delta t)r_{1\perp} = \frac{1}{2}(v_2\Delta t)r_{2\perp} \quad (22)$$

Dividing Equation 22 through by Δt and multiplying both sides by $2m$, where m is the mass of the satellite, gives

$$m_1v_1r_{1\perp} = m_2v_2r_{2\perp} \quad (23)$$

Recall that the definition of a particle’s angular momentum about some axis is the linear momentum $\vec{p} = m\vec{v}$ times the lever arm r_{\perp} (see Equations 7–15, 16). Thus the left side of Equation 23 is the satellite’s angular momentum at position 1, the right side at position 2. The statement that the satellite sweeps out equal areas in equal times is thus equivalent to the statement that the satellite’s angular momentum mvr_{\perp} has the same value all around the orbit. Like the dumbbell in Figure (34), the satellite moves faster when r_{\perp} is small, and slower when r_{\perp} is large, in order to conserve angular momentum.

As a direct check of the conservation of angular momentum in the satellite orbit program, note that if a particle is located a distance x from an axis of rotation and is moving in the y direction with a velocity v_y as shown in Figure (36a), the lever arm about the origin is simply x , and the particle’s angular momentum about the origin ℓ_a is

$$\ell_a = mxv_y \quad \text{particle's angular momentum in Figure (36a)} \quad (24)$$

Using the right hand convention illustrated in Figure (7-14), we see that this particle has angular momentum directed up, out of the paper. We will call this positive angular momentum. (You can think of m as a small piece of the bicycle wheel shown in Figure 7-14.)

Now consider a particle of mass m located a distance y from the origin traveling in the $-x$ direction as shown in Figure (36b). By the right hand convention the angular momentum is still positive (you could think of this m as another part of the same bicycle wheel), but the x velocity is now negative. Thus the formula for this particle’s angular momentum is

$$\ell_b = -myv_x \quad (25)$$

We have to put in the minus ($-$) sign to counteract the fact that v_x is negative but ℓ_b is positive.

It turns out that if a particle is in the xy plane at some arbitrary position $\vec{R} = (x,y)$, and has some arbitrary velocity $\vec{v} = (v_x,v_y)$ in the xy plane, then the formula for the angular momentum ℓ_0 of the particle about the origin is

$$\ell_0 = m(xv_y - yv_x) \quad (26)$$

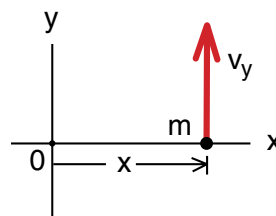


Figure 36a
Here $\ell = mxv_y$.

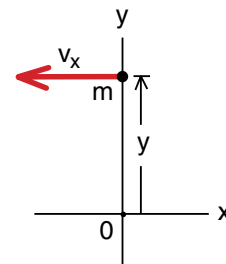


Figure 36b
Here $\ell = myv_x$.

You can see that this general result is just a combination of the two special cases we considered in Figures (36) and Equations 24 and 25. (Equation 26 also comes from the formula $\vec{l} = m \vec{r} \times \vec{v}$ where $\vec{r} \times \vec{v}$ is the vector cross product of \vec{r} and \vec{v} . We will discuss vector cross products in detail later in Chapter 11. For now Equation 26 is all we need.)

With Equation 26, we can easily test whether angular momentum is in fact conserved in our satellite orbit calculations. By the end of the calculational loop, we have already calculated new values of the satellite's x and y coordinates R_x and R_y , and x and y velocity components v_x and v_y . Thus to calculate the satellite's angular momentum, all we need is the line

$$\text{LET } L_z = M * (R_x * V_y - R_y * V_x) \quad (27)$$

where we are using the name L_z because we are observing the z component of the satellite's angular momentum, as indicated in Figure (37).

To check that angular momentum is conserved, we could add a print line at the end of the calculational loop like

$$\text{IF MOD (I, 40) = 0 THEN PRINT } R_x, R_y, L_z \quad (28)$$

By printing the values of R_x and R_y as well as L_z , we can see where the satellite is in its orbit as well as the value of the angular momentum at that point.

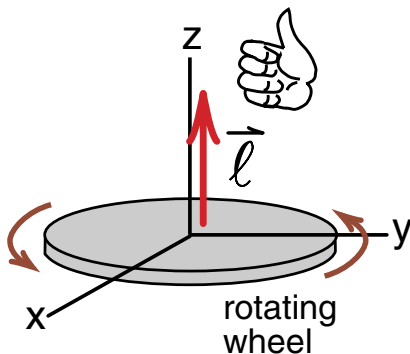


Figure 37
Angular momentum vector of a rotating wheel.

Exercise 19

Add lines (27) and (28) to your Orbit 1 program and check that angular momentum is conserved. Use several different initial conditions so that you can check conservation of angular momentum for different elliptical orbits. (Make sure that L_z is calculated within the calculational loop so that the latest values of R_x , R_y , V_x and V_y are used for each calculation.) Also, if you set the satellite mass m equal to 1, the values for L_z will be easier to interpret. (The value of the constant m does not matter since you are simply checking that L_z is constant during the satellite's orbital motion.)

Exercise 20

The fact that angular momentum is conserved in Exercise 19 should not be too surprising because you have already checked in earlier exercises that the elliptical orbit obeys Kepler's second law, and as we have just seen, Kepler's second law implies conservation of angular momentum. In this exercise, see if angular momentum is also conserved if we modify the gravitational force law as we did in Exercise 19. Take your program from Exercise 19, the one that prints out the values of the angular momentum, change the exponent of r in Newton's law of gravity, and see if angular momentum is conserved while the ellipse is precessing.

CONSERVATION OF ENERGY

In addition to angular momentum, there is another quantity that is conserved during a satellite's orbital motion. In Chapter 10, which is completely devoted to the topic of energy, we will discuss techniques for deriving formulas for various forms of energy. But it is not necessary to be able to derive energy formulas in order to be able to appreciate and use the concept.

The fundamental idea behind the concept of energy is that energy is a conserved quantity. To study the conservation of energy is often a more difficult job than studying the conservation of linear or angular momentum, because there are many forms that energy can take, and not all the forms are easy to recognize. But in certain simple examples like the motion of an earth satellite, there are only two forms of energy we have to deal with, and the conservation of energy is easy to observe.

Unlike linear and angular momentum, energy does not point anywhere. Energy is represented by a number, not a vector. You get a bill from your electric company for the amount of electrical energy you used the previous month. The electric company has a formula, based on the reading of your electric meter, for the amount of electrical energy you used. Because energy is conserved, the power company could not create the energy they sold you out of nothing, they probably got the energy either from a nuclear power plant or by burning fossil fuels. If they got the energy from fossil fuels, that energy originally came from the sun, from the combining of hydrogen nuclei to form helium nuclei. If the electricity came from a nuclear power plant, the energy came from the splitting of large uranium or plutonium nuclei into smaller nuclei. The uranium and plutonium nuclei were formed by getting their energy from a supernova explosion that must have occurred over five billion years ago.

In our discussion of energy in Chapter 10, we will see that there is a close analogy between keeping track of your checkbook balance in a bank and keeping a record of the amount of energy a system has. With a bank

balance, there is a convention that if your balance is positive, the bank owes you money, and if the balance is negative, you owe the bank money. A zero balance indicates that neither owes each other anything. If the bank is not worried about your credit, it does not make much difference whether your balance is positive, negative or zero, you can still write checks, make deposits, and go about your normal business.

In the way we deal with energy, what we call the zero of energy does not make much difference either. We can think of a power company borrowing energy from a coal company just as it borrows money from a bank. In this sense the power company can have a negative energy balance just as it has a negative bank balance. The fact that energy is conserved means that the power company cannot create energy out of nothing to repay the debt. The difference between the power company and physical systems like satellites in orbit is that we let power companies pay their energy debt with cash, a physical system can increase its energy balance only by getting energy from somewhere else.

In our accounting scheme for energy, some terms are positive and some are negative. The term called *kinetic energy* is always positive. In most circumstances, kinetic energy is given by the formula $1/2 mv^2$ where m is the mass of the object and v the object's speed. Kinetic energy is positive because neither m or $1/2 mv^2$ can become negative.

To observe conservation of energy for satellite motion, it is necessary to account for two forms of energy. One is kinetic energy $1/2v^2$, the other is what is called *gravitational potential energy*. Our formula for gravitational potential energy will be $-Gm_s m_e / r$ where G is the gravitational constant, m_s and m_e the masses of the satellite and earth respectively, and r the separation between them. This formula looks much like the gravitational force formula, except that it is proportional to $1/r$ rather than $1/r^2$.

What is often upsetting to students when they first encounter the gravitational potential energy formula is the minus sign. How can energy be negative? This is essentially a result of our accounting procedure. The important feature of energy is that it is conserved. If the gravitational potential energy in some part of an orbit becomes more negative, then the kinetic energy has to become more positive so that the total is conserved, i.e., stays constant. As far as energy conservation is concerned, it does not make any difference what the total energy is, as long as it is constant.

At this point we have made no effort to explain where the formulas $\frac{1}{2}mv^2$ for kinetic energy and $-Gm_s m_e/r$ for gravitational potential energy came from. That is a subject for Chapter 10. What we are concerned with now is to see if the **Total Energy**, the sum of these two, is conserved as the satellite moves around its orbit.

$$\begin{aligned} \text{total energy} \\ \text{of a satellite} \\ \text{in orbit} \end{aligned} = \begin{aligned} \text{kinetic} \\ \text{energy} \end{aligned} + \begin{aligned} \text{gravitational} \\ \text{potential} \\ \text{energy} \end{aligned} \quad (29)$$

$$E_{\text{tot}} = \frac{1}{2}mv^2 - \frac{Gm_s m_e}{r}$$

We will check for conservation of energy in much the same way we checked for conservation of angular momentum using our Orbit 1 program. Near the end of the calculational loop, after we have calculated the latest values of the satellite position \vec{r} and velocity \vec{v} , and have also calculated the corresponding magnitudes r and v , we can add the line

$$\text{LET Etot} = \text{Ms} * \text{V} * \text{V} / 2 - \text{G} * \text{Ms} * \text{Me} / \text{R} \quad (30)$$

Then we can add a print line like

$$\text{IF MOD(I, 40)} = 0 \text{ THEN PRINT Rx, Ry, Etot}$$

By looking at the printed values of E_{tot} we can see whether this formula for E_{tot} is conserved as the satellite moves around.

Exercise 21

Using the steps described above, check that the satellite's total energy E_{tot} is conserved. (You will notice slight variations in the value of E_{tot} ; the values are not as steady as they were in the printout of angular momentum. Exercise 22 suggests a way of improving the energy calculation and getting better results.)

As a variation, print out the values of the kinetic energy, potential energy and E_{tot} . You will see big changes in the kinetic and potential energy, while the sum E_{tot} remains nearly constant. Start the satellite with different initial conditions and check for energy conservation for different elliptical orbits.

Exercise 22

We can obtain a more accurate calculation of the satellite's total energy by slightly modifying the value of v used in the kinetic energy formula. When we put the calculation of E_{tot} at the end of the calculational loop, we are using the value of v at the end of the time step dt . It turns out that we get a more accurate energy calculation if we use a value of v that is the average of the value we had when we entered the calculational loop and the value a time dt later when we left. This averaging is easily accomplished using the following commands inserted into your calculational loop.

```
LET Vold = V           new line saving old value of v
LET Vx = ...
LET Vy = ...          your old lines calculating
                       the next new value of v
LET V = SQR (Vx * Vx + Vy * Vy)
LET Vnew = V          saving the new value of V
LET V = (Vold + Vnew) / 2  setting V to the average value
LET Etot = (Ms * V * V) / 2 - G * Ms * Me / R
```

The steps above using the average of V_{new} and V_{old} for V in the calculation of the kinetic energy represents the kind of specialized computer trick we have tried to avoid in this text. However, the trick works so well, the improvement in the value of the total energy is so great that it is worth the effort. This is particularly true for project work where a check for conservation of energy is the main check of the validity of the calculation. (You can usually spot computer errors by printing out the total energy, because computer errors almost never conserve energy.)

Exercise 23 (optional, more like a project)

It turns out that if we modify the formula for the gravitational force, for example changing the exponent of r from $+2$ to -1.9 , we also have to modify the formula for the gravitational potential energy in order to observe energy conservation. You will learn in Chapter 10 that the formula for the gravitational potential energy is the integral of the magnitude of the force. We can, for example, obtain our formula for gravitational potential energy from the gravitational force formula by the following integration

$$\int_{\infty}^r \frac{Gm_s m_e}{r^2} dr = -\frac{Gm_s m_e}{r} \quad (31)$$

If you modify the gravitational force formula, you can do the same kind of integration to get the corresponding potential energy formula. (In Chapter 10 we will have a lot more to say about this integration. For now you can treat the integration as a convenient device for obtaining the potential energy formula. Since the important feature of energy is that it is conserved, if you find from running your Orbit 1 program that the total energy turns out to be conserved, you know you have the correct potential energy formula no matter how it was derived.)

For this exercise, start by modifying the gravitational force law by changing the exponent of r from $+2$ to -1.9 . Then run your Orbit 1 program using the formula $-Gm_s m_e/r$ for potential energy to see that this formula does not work. (Use the accurate version of the program from Exercise 22 so that you can be more confident of the results.)

Then integrate $Gm_s m_e/r^{1.9}$ to find a new potential energy formula. See if energy is conserved with your new formula. Once this is successful, try some other modification.

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