Our discussion of rotational motion begins with a review of the measurement of angles using the concept of radians. We will refer to an angle measured in radians as an **angular distance**. If we are discussing an object that is rotating, we will describe the rotation in terms of the increase in angular distance, namely an **angular velocity**. And if the speed of rotation is changing, we will describe the change in terms of an **angular acceleration**.

In Chapter 7, linear momentum and angular momentum were treated as distinctly separate topics. The main point of this chapter is to develop a close analogy between the two concepts. The linear momentum of an object is its mass \( m \) times its linear velocity \( v \). We will see that angular momentum can be expressed as an angular mass times an angular velocity. (Angular mass is more commonly known as moment of inertia). Then, using the formalism of the vector cross product (mentioned in Chapter 2), we will see that angular momentum can be treated as a vector quantity, which explains the bicycle wheel experiments we discussed in Chapter 7.

The fundamental concept of Newtonian mechanics is that the total force \( \vec{F} \) acting on an object is equal to the time rate of change of the object’s linear momentum; \( \vec{F} = \frac{d\vec{p}}{dt} \). Using the vector cross product formalism, we will obtain a complete angular analogy to this equation. We will find that a quantity we call an angular force is equal to the time rate of change of angular momentum. (The angular force is more commonly known as torque).

The angular analogy to Newton’s second law looks a bit peculiar at first. It involves lever arms and vectors that point in funny directions. After some demonstrations to show that the equation appears to give reasonable results, we apply the equation to predict the motion of a gyroscope. The prediction appears to be absurd, but we find that that is the way a gyroscope behaves.

Our focus in this chapter is on angular momentum because that concept will play such an important role in our later discussions of atomic physics and electrons and nuclear magnetic resonance. There are other important and interesting topics such as rotational kinetic energy and the calculation of moments of inertia which we discuss in more detail in the appendix. These topics are not difficult and lead to some good lecture demonstrations and laboratory experiments. We put them in an appendix because they do not play the essential role that angular momentum does in our later discussions.
**Radian Measure**

From the point of view of doing calculations, it is more convenient to measure an angle in radians than the more familiar degrees. In radian measure the angle \( \theta \) shown in Figure (1) is the ratio of the arc length \( s \) to the radius \( r \) of the circle

\[
\theta \equiv \frac{s}{r} \text{ radians} \tag{1}
\]

Since \( s \) and \( r \) are both distances, the ratio \( s/r \) is a dimensionless quantity. However we will find it convenient for the angular analogy to keep the name radians as if it were the actual dimension of the angle. For example we will measure angular velocities in \textit{radians per second}, which is analogous to linear velocities measured in meters per second.

Since the circumference of a circle is \( 2\pi r \), the number of radians in a complete circle is

\[
\theta \text{ (complete circle)} = \frac{2\pi r}{r} = 2\pi \]

In discussing rotation, we will often refer to going around one complete time as one complete \textit{cycle}. In one cycle, the angle \( \theta \) increases by \( 2\pi \). Thus \( 2\pi \) is the number of \textit{radians per cycle}. We will find it convenient to assign these dimensions to the number \( 2\pi \):

\[
\frac{2\pi \text{ radians}}{\text{cycle}} \tag{2}
\]

To relate radians to degrees, we use the fact that there are 360 degrees/cycle and dimensional analysis to find the number of degrees/radian

\[
\frac{360 \text{ degrees}}{\text{cycle}} \times \frac{1}{2\pi \text{ radians}} = \frac{360}{2\pi} \text{ degrees/radian} = 57.3 \text{ degrees/radian} \tag{3}
\]

Fifty seven degrees is a fairly awkward unit angle for purposes of drafting and navigation; no one in his or her right mind would mark a compass in radians. However, in working with the dynamics of rotational motion, radian measure is the only reasonable choice.

**Angular Velocity**

The typical measure of angular velocity you may be familiar with is revolutions per minute (RPM). The tachometer in a sports car is calibrated in RPM; a typical sports car engine gives its maximum power around 5000 RPM. Engine manufacturers in Europe are beginning to change over to revolutions per second (RPS), but somehow revving an engine up to 83 RPS doesn't sound as impressive as 5000 RPM. (Tachometers will probably be calibrated in RPM for a while.) In physics texts, angular velocity is measured in radians per second. Since there are \( 2\pi \) radians/cycle, 83 revolutions or cycles per second corresponds to \( 2\pi \times 83 = 524 \) radians/second. Few people would know what you were talking about if you said that you should shift gears when the engine got up to 524 radians per second.

---

**Exercise 1**

What is the angular velocity, in radians per second, of the hour hand on a clock?

---

**Figure 1**

The angle \( \theta \) in radians is defined as the ratio of the arc length \( s \) to the radius \( r \): \( \theta = s/r \).
Our formal definition of angular velocity is the time rate of change of an angle. We almost always use the Greek letter \( \omega \) (omega) to designate angular velocity

\[
\text{angular velocity} \quad \omega = \frac{d\theta}{dt} \text{ radians} \quad \text{second} \quad (4)
\]

When thinking of angular velocity \( \omega \) picture a line marked on the end of a rotating shaft. The angle \( \theta \) is the angle that the line makes with the horizontal as shown in Figure (2). As the shaft rotates, the angle \( \theta(t) \) increases with time, increasing by \( 2\pi \) every time the shaft goes all the way around.

**Angular Acceleration**

When we start a motor, the angular velocity of the shaft starts at \( \omega = 0 \) and increases until the motor gets up to its normal speed. During this start-up, \( \omega(t) \) changes with time, and we have an angular acceleration \( \alpha \) defined by

\[
\text{angular acceleration} \quad \alpha = \frac{d\omega}{dt} \text{ radians} \quad \text{second}^2 \quad (5)
\]

The angular acceleration \( \alpha \) has the dimensions of radians/sec\(^2\) since the derivative gives us another factor of time in the denominator. Combining Equation 4 and 5 relates \( \alpha \) to \( \theta \) by

\[
\alpha = \frac{d^2\theta}{dt^2} \quad (6)
\]

**Angular Analogy**

At this point we have a complete analogy between the rotation of a motor shaft and one dimensional linear motion. This analogy becomes clear when we write out the definitions of position, velocity, and acceleration:

<table>
<thead>
<tr>
<th>Linear motion</th>
<th>Angular motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>( x ) meters  ( \theta ) radians</td>
</tr>
<tr>
<td>Velocity</td>
<td>( v = \frac{dx}{dt} ) meters/second ( \omega = \frac{d\theta}{dt} ) radians/second</td>
</tr>
<tr>
<td>Acceleration</td>
<td>( a = \frac{dv}{dt} ) meters/second(^2) ( \alpha = \frac{d\omega}{dt} ) radians/second(^2)</td>
</tr>
</tbody>
</table>

As far as these equations go, the analogy is precise. Therefore any formulas that we derived for linear motion in one dimension must also apply to angular motion. In particular the constant acceleration formulas, derived in Chapter 3, must apply. If the linear and angular accelerations \( a \) and \( \alpha \) are constant, then we get

\[
x = v_0 t + \frac{1}{2} a t^2 \quad \theta = \omega_0 t + \frac{1}{2} \alpha t^2 \quad (8)
\]

\[
v = v_0 + a t \quad \omega = \omega_0 + \alpha t \quad (9)
\]

**Exercise 2**

An electric motor, that turns at 3600 rpm (revolutions per minute) gets up to speed in 1/2 second. Assume that the angular acceleration \( \alpha \) was constant while the motor was getting up to speed.

a) What was \( \alpha \) (in radians/sec\(^2\))?

b) How many radians, and how many complete cycles, did the shaft turn while getting up to speed?
**Tangential Distance, Velocity and Acceleration**

So far we have used the model of a rotating shaft to illustrate the concepts of angular distance, velocity and acceleration. We now wish to shift the focus of our discussion to the dynamics of a particle traveling along a circular path. For this we will use the model of a small mass \( m \) on the end of a massless stick of length \( r \) shown in Figure (3). The other end of the stick is attached to and is free to rotate about a fixed axis at the origin of our coordinate system. The presence of the stick ensures that the mass \( m \) travels only along a circular path of radius \( r \). The quantity \( \theta(t) \) is the angular distance travelled and \( \omega(t) \) the angular velocity of the particle.

When we are discussing the motion of a particle in a circular orbit, we often want to know how far the particle has travelled, or how fast it is moving. The distance \( s \) along the path (we could call the tangential distance) travelled is given by Equation 1 as

\[
s = r\theta
\]

(10)

*Figure 3*

*Mass rotating on the end of a massless stick.*

The speed of the particle along the path, which we can call the tangential speed \( v_t \), is the time derivative of the tangential distance \( s(t) \)

\[
v_t = \frac{ds(t)}{dt} = \frac{d[r\theta(t)]}{dt} = r \frac{d\theta(t)}{dt}
\]

where \( r \) comes outside the derivative since it is constant. Since \( d\theta(t)/dt \) is the angular velocity \( \omega \), we get

\[
v_t = r\omega
\]

(11)

*Figure 4*

*Particle moving at a constant speed in a circle of radius \( r \) accelerates toward the center of the circle with an acceleration of magnitude \( a_r = v^2/r \).*

The tangential acceleration \( a_t \), the acceleration of the particle along its path, is the time derivative of the tangential velocity

\[
a_t = \frac{dv_t(t)}{dt} = \frac{d[r\omega(t)]}{dt} = r \frac{d\omega(t)}{dt} = r\alpha
\]

(12)

where again we took the constant \( r \) outside the derivative, and used \( \alpha = d\omega/dt \).
Radial Acceleration

If the angular velocity $\omega$ is constant, if we have a particle traveling at constant speed in a circle, then $\alpha = \frac{d\omega}{dt} = 0$ and there is no tangential acceleration $a_t$. However, we have known from almost the beginning of the course that a particle traveling at constant speed $v$ in a circle of radius $r$ has an acceleration directed toward the center of the circle, of magnitude $v^2/r$, as shown in Figure (4). We will now call this center directed acceleration the radial acceleration $a_r$.

$$a_r = \frac{v_t^2}{r} \quad \text{radial acceleration}$$

(13)

Exercise 3

Express the radial acceleration $a_r$ in terms of the orbital radius $r$ and the particle’s angular velocity $\omega$.

If a particle is traveling in a circular orbit, but its speed $v_t$ is not constant, then it has both a radial acceleration $a_r = \frac{v_t^2}{r}$, and a tangential acceleration $a_t = r\alpha$. The radial acceleration is always directed toward the center of the circle and always has a magnitude $v^2/r$. The tangential acceleration, if it exists, is tangential to the circle, pointing forward (counterclockwise) if $\alpha$ is positive and backward if $\alpha$ is negative. These accelerations are shown in Figure (5).

Figure 5
Motion with radial and tangential acceleration.

Bicycle Wheel

For much of the remainder of the chapter, we will use a bicycle wheel, often weighted with wire wound around the rim, to illustrate various phenomena of rotational motion. Conceptually we can think of the bicycle wheel as a collection of masses on the ends of massless rods as shown in Figure (6). The massless rods form the spokes of the wheel, and we can think of the masses $m$ as fusing together to form the wheel. When forming a wheel, all the masses have the same radius $r$, same angular velocity $\omega$ and same angular acceleration $\alpha$. If we choose one point on the wheel from which to measure the angular distance $\theta$, then as far as angular motion is concerned, it does not make any difference whether we are discussing the mass on the end of a rod shown in Figure (3) or the bicycle wheel shown in Figure (6). Which model we use depends upon which provides a clearer insight into the phenomena being discussed.

Figure 6
Bicycle wheel as a collection of masses on the end of massless rods.
ANGULAR MOMENTUM

In Chapter 7, we defined the angular momentum \( \ell \) of a mass \( m \) traveling at a speed \( v \) in a circle of radius \( r \) as

\[ \ell = mvr \]  

(7-11)

As we saw, in Figure (7-9) reproduced here, the quantity \( \ell = mvr \) did not change when we had a ball moving in a circle on the end of a string, and we pulled in on the string. The radius of the circle decreased, but the speed increased to keep the product \( vr \) constant. This was our introduction to the concept of the conservation of angular momentum.

After that, we went on to consider some rather interesting experiments where we held a rotating bicycle wheel while standing on a freely turning platform. We found that these experiments could be explained qualitatively if we thought of the angular momentum of the bicycle wheel as being a vector quantity which pointed along the axis of the wheel, as shown in Figure (7-15) reproduced below. What we will do now is develop the formalism which treats angular momentum as a vector.

Angular Momentum of a Bicycle Wheel

We will begin our discussion of the angular momentum of a bicycle wheel using the picture of a bicycle wheel shown in Figure (6), i.e., a collection of balls on the end of massless rods or spokes. If the wheel is rotating with an angular velocity \( \omega \), then each ball has a tangential velocity \( v_t \) given by Equation 11a

\[ v_t = r\omega \]  

(11 repeated)

If the \( i \)-th ball in the wheel (identified in Figure 7) has a mass \( m_i \), then its angular momentum \( \ell_i \) will be given by

\[ \ell_i = m_i v_t r = m_i (r\omega) r \]

\[ \ell_i = (m_i r^2) \omega \]  

(14)

Assuming that the total angular momentum \( L \) of the bicycle wheel is the sum of the angular momenta of each ball (we will discuss this assumption in more detail shortly) we get

\[ L = \sum_i \ell_i = \sum_i (m_i r^2 \omega) \]  

(15)

Since each mass \( m_i \) is at the same radius \( r \) and is traveling with the same angular velocity \( \omega \), we get

\[ L = \left( \sum_i m_i \right) r^2 \omega \]

Noting that \( M = \sum_i m_i \) is the total mass of the bicycle wheel, we get

\[ L = Mr^2 \omega \]  

angular momentum of a bicycle wheel  

(16)

Figure 7-9
Ball on the end of a string, swinging in a circle.

Figure 7-15
When the bicycle wheel is turned over and its angular momentum points down, the person starts rotating with twice as much angular momentum, pointing up.

Figure 7
The angular momentum of the \( i \)-th ball is \( m_i v_t r_i \).
Angular Velocity as a Vector

To explain the bicycle wheel experiment discussed in Chapter 7, we assumed that the angular momentum \( \mathbf{L} \) was a vector pointing along the axis of the wheel as shown in Figure (8a). We can obtain this vector concept of angular momentum by first defining a vector angular velocity \( \mathbf{\omega} \) as shown in Figure (8b). We will say that if a wheel is rotating with an angular velocity \( \omega \text{ rad/sec} \), the vector \( \mathbf{\omega} \) has a magnitude of \( \omega \text{ rad/sec} \), and points along the axis of rotation as shown in Figure (8b). Since the axis has two directions, we use a right hand convention to select among them. Curl the fingers of your right hand in the direction of the direction of the rotation, and the thumb of your right hand will point in the direction of the vector \( \mathbf{\omega} \).

Angular Momentum as a Vector

Since the vector \( \mathbf{\omega} \) points in the direction we want the angular momentum vector \( \mathbf{L} \) to point, we can obtain a vector formula for \( \mathbf{L} \) by simply replacing \( \omega \) by \( \mathbf{\omega} \) in Equation 16 for the angular momentum of the bicycle wheel

\[
\mathbf{L} = (\text{Mr}^2) \mathbf{\omega}
\]

(17)  

Figure 8a
The angular momentum vector.

Figure 8b
The angular velocity vector.

ANGULAR MASS OR MOMENT OF INERTIA

Equation 17 expresses the angular momentum \( \mathbf{L} \) of a bicycle wheel as a numerical quantity \( (\text{Mr}^2) \) times the vector angular velocity \( \mathbf{\omega} \). This is not very different from linear momentum \( \mathbf{p} \) which is the mass (M) times the linear velocity vector \( \mathbf{v} \)

\[
\mathbf{p} = M\mathbf{v}
\]

(18)

We obtain an analogy between linear and angular momentum if we call the quantity \( (\text{Mr}^2) \) the angular mass of the bicycle wheel. Designating the angular mass by the letter I, we get

\[
\mathbf{L} = I\mathbf{\omega}
\]

(19)  

\[
I = \text{Mr}^2
\]

(20)

The quantity I is usually called moment of inertia rather than angular mass, but angular mass provides a better description of what we are dealing with. We will use either name, depending upon which seems more appropriate.
Calculating Moments of Inertia

Equation 20 is not the most general formula for calculating moments of inertia. The bicycle wheel is special in that all the mass is essentially out at a single radius \( r \). If, instead, we had a solid wheel where the mass was spread out over different radii, we would have to conceptually break the wheel into a number of separate rim-like wheels of radii \( r_i \) and mass \( m_i \), calculate the moment of inertia of each rim, and add the results together to get the total moment of inertia.

In Appendix A we have a relatively complete discussion of how to calculate moments of inertia, and how moment of inertia is related to rotational kinetic energy. There you will see that rotational kinetic energy is \( \frac{1}{2} I \omega^2 \), which is analogous to the linear kinetic energy \( \frac{1}{2} M v^2 \). This material is placed in an appendix, not because it is difficult, but because we do not wish to digress from our discussion of the analogy between linear and angular momentum. At this point, one example and one exercise should be a sufficient introduction to the concept of moment of inertia.

Example 1

Calculate the moment of inertia, about its axis, of a cylinder of mass \( M \) and outside radius \( R \). Assume that the cylinder has uniform density.

Solution: We conceptually break the cylinder into a series of concentric cylinders of radius \( r \) and thickness \( dr \) as shown in Figure (9). Each hollow cylinder has a mass given by

\[
dm = M \times \frac{\text{end area of hollow cylinder}}{\text{total end area}}
\]

\[
= M \times \frac{2 \pi r \, dr}{\pi R^2} = M \times \frac{2r \, dr}{R^2}
\]  \hspace{1cm} (21)

Since all the mass in the hollow cylinder is out at a radius \( r \), just as it is for a bicycle wheel, the hollow cylinder has a moment of inertia \( dI \) given by

\[
dI = (dm) \times r^2
\]

\[
= M \times \frac{2r \, dr}{R^2} \times r^2 = \frac{2Mr^3 \, dr}{R^2}
\]  \hspace{1cm} (22)

The total moment of inertia of the cylinder is the sum of the moments of inertia of all the hollow cylinders. This addition is done by integrating the formula for \( dI \) from \( r = 0 \) out to \( r = R \).

\[
I_{\text{solid cylinder}} = \int_{r=0}^{r=R} dI
\]

\[
= \int_{r=0}^{r=R} \frac{2Mr^3 \, dr}{R^2}
\]

\[
= \frac{2M}{R^2} \int_{r=0}^{r=R} r^3 \, dr
\]

\[
= \frac{2M}{R^2} \left[ \frac{r^4}{4} \right]_0^R = \frac{2MR^4}{4R^2}
\]  \hspace{1cm} (23)

\[
I_{\text{solid cylinder}} = \frac{1}{2} MR^2
\]

Figure 9

Calculating the moment of inertia of a cylinder about its axis of rotation.
Two points are made in Example 1. The first is that calculating the moment of inertia of an object usually requires an integration, because different parts of the object are out at different distances \( r \) from the axis of rotation. Secondly we see that the moment of inertia of a solid cylinder is less than the moment of inertia of a bicycle wheel of the same mass and outer radius \((1/2 MR^2\) for the cylinder versus \(MR^2\) for the bicycle wheel). This is because all the mass of the bicycle wheel is out at the maximum radius \( R \), while most of the mass of the solid cylinder is in at smaller radii.

A considerable amount of time can be spent discussing the calculation of moments of inertia of various shaped objects. Rather than do that here, we will simply present a table of the moments of inertia of common objects of mass \( M \) and outer radius \( R \), about an axis that passes through the center.

<table>
<thead>
<tr>
<th>Object</th>
<th>Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>cylindrical shell</td>
<td>( 1 ) ( MR^2 )</td>
</tr>
<tr>
<td>solid cylinder</td>
<td>( 1/2 ) ( MR^2 )</td>
</tr>
<tr>
<td>spherical shell</td>
<td>( 2/3 ) ( MR^2 )</td>
</tr>
<tr>
<td>solid sphere</td>
<td>( 2/5 ) ( MR^2 )</td>
</tr>
</tbody>
</table>

**Exercise 4**

As shown in Figure (10) we have a thick-walled hollow brass cylinder of mass \( M \), with an inner radius \( R_i \) and outer radius \( R_o \). Calculate its moment of inertia about its axis of symmetry. Check your answer for the case \( R_i = 0 \) (a solid cylinder) and for \( R_i = R_o \) (which corresponds to the bicycle wheel).

---

**VECTOR CROSS PRODUCT**

The idea of having the angular velocity \( \bar{\omega} \) being a vector pointing along the axis of rotation gave us a nice analogy between linear momentum \( \bar{p} = M \bar{v} \) and angular momentum \( \bar{L} = I \bar{\omega} \). But to obtain the dynamical equation for angular momentum, the one analogous to Newton’s second law for linear momentum, we need the mathematical formalism of the vector cross product defined back in Chapter 2. Since we have not used the vector cross product before now, we will briefly review the topic here.

If we have two vectors \( \bar{A} \) and \( \bar{B} \) like those shown in Figure (11), the vector cross product \( \bar{A} \times \bar{B} \) is defined to have a magnitude

\[
| \bar{A} \times \bar{B} | = |\bar{A}| |\bar{B}| \sin \theta
\]

(24)

where \( |\bar{A}| \) and \( |\bar{B}| \) are the magnitudes of the vectors \( \bar{A} \) and \( \bar{B} \), and \( \theta \) is the small angle between them. Note that when the vectors are parallel, \( \sin \theta = 0 \) and the cross product is zero. The cross product is a maximum when the vectors are perpendicular. This is just the opposite from the scalar dot product which is a maximum when the vectors are parallel and zero when perpendicular. Conceptually you can think of the dot product as measuring parallelism while the cross product measures perpendicularity.

The other major difference between the dot and cross product is that with the dot product we end up with a number (a scalar), while with the cross product, we end up with a vector. The direction of \( \bar{A} \times \bar{B} \) is the most peculiar feature of the cross product; it is **perpendicular** to the plane defined by the vectors \( \bar{A} \) and \( \bar{B} \). If we draw \( \bar{A} \) and \( \bar{B} \) on a sheet of paper as we did in Figure (11), then the directions perpendicular to both \( \bar{A} \) and \( \bar{B} \) are either up out of the paper or down into the paper. To decide which of these two directions to choose, we use the following right hand rule. (This is an arbitrary convention, but if you use it consistently in all of your calculations, everything works out OK).

---

**Figure 10**

*Thick-walled hollow cylinder.*

**Figure 11**

*The vectors \( \bar{A} \) and \( \bar{B} \).*
Right Hand Rule for Cross Products
To find the direction of the vector $\mathbf{A} \times \mathbf{B}$, point the fingers of your right hand in the direction of the first vector in the product (namely $\mathbf{A}$). Then, without breaking your knuckles, curl the finger of your right hand toward the second vector $\mathbf{B}$. Curl them in the direction of the small angle $\theta$. If you do this correctly, the thumb of your right hand will point in the direction of the cross product $\mathbf{A} \times \mathbf{B}$. Applying this to the vectors in Figure (11), we find that the vector $\mathbf{A} \times \mathbf{B}$ points up out of the paper as shown in Figure (12).

Exercise 5
(a) Follow the steps we just mentioned to show that $\mathbf{A} \times \mathbf{B}$ from Figure (11) does point up out of the paper.

(b) Show that the vector $\mathbf{B} \times \mathbf{A}$ points down into the paper.

If you did the exercise (5b) correctly, you found that $\mathbf{B} \times \mathbf{A}$ points in the opposite direction from the vector $\mathbf{A} \times \mathbf{B}$. In all previous examples of multiplication you have likely to have encountered, the order in which you did the multiplication made no difference. For example, both $3 \times 5$ and $5 \times 3$ give the same answer 15. But now we find that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$ and the order of the multiplication does make a difference. Mathematicians say that cross product multiplication does not commute.

There is one other special feature of the cross product worth noting. If $\mathbf{A}$ and $\mathbf{B}$ are parallel, or anti parallel, then they do not define a unique plane and there is no unique direction perpendicular to both of them. Various possibilities are indicated in Figure (13). But when the vectors are parallel or anti parallel, $\sin \theta = 0$ and the cross product is zero. The special case where the cross product does not have a unique direction is when the cross product has zero magnitude with the result that the lack of uniqueness does not cause a problem.

Figure 13
If the vectors $\mathbf{A}$ and $\mathbf{B}$ are either parallel or antiparallel, then as shown above, there is a whole plane of vectors perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.
**CROSS PRODUCT DEFINITION OF ANGULAR MOMENTUM**

Let us now see how we can use the idea of a vector cross product to obtain a definition of angular momentum vectors. To explain the bicycle wheel experiments, we wanted the angular momentum to point along the axis of the wheel as shown in Figure (14a). Since there are two directions along the axis, we have arbitrarily chosen the direction defined by the right hand convention shown. (Curl the fingers of your right hand in the direction of the rotation and your thumb will point in the direction of \( \vec{L} \)).

**Figure 14a**
*Right hand rule for angular momentum.*

In Figure (14b) we went to the masses and spoke model of the bicycle wheel, and selected one particular mass which we called \( m_i \). This mass is located at a coordinate vector \( \vec{r}_i \) from the center of the wheel, and is traveling with a velocity \( \vec{v}_i \). According to our definition of angular momentum in Chapter 7, using the formula \( \vec{L} = mvr \), the ball’s angular momentum should be

\[
\vec{L} = m_i \vec{r}_i \vec{v}_i \tag{7-11 again}
\]

What we want to do now is to turn this definition of angular momentum into a vector that points down the axis of the wheel. This we can do with the vector cross product of \( \vec{r}_i \) and \( \vec{v}_i \). We will try the definition of the vector \( \vec{l}_i \) as

\[
\vec{l}_i = m_i (\vec{r}_i \times \vec{v}_i) \tag{25}
\]

**Figure 14b**
*Angular momentum of one of the balls in the ball-spoke model of a bicycle wheel.*

**Exercise 6**

a) Look at Figure (14c) showing the vectors \( \vec{r}_i \) (which point into the paper) and \( \vec{v}_i \). Point the fingers of your right hand in the direction of \( \vec{r}_i \) and then curl them toward the vector \( \vec{v}_i \). Does your thumb point in the direction of \( \vec{l}_i \) shown? (If it does not, you have peculiar knuckle joints or are not following instructions).

**Figure 14c**
*The three vectors \( \vec{r}_i, \vec{v}_i \) and \( \vec{l}_i \).*

b) Choose any other mass that forms the bicycle wheel shown in Figure (14b). Call that the mass \( m_i \). Show that the vector \( \vec{l}_i = m_i (\vec{r}_i \times \vec{v}_i) \) also points down the axis, parallel to \( \vec{l}_i \). Try this for several different masses, say one at the top, one at the front, and one at the bottom of the wheel.

If you did Exercise 6 correctly, you found that all the angular momentum vectors \( \vec{l}_i = m_i (\vec{r}_i \times \vec{v}_i) \) were parallel to each other, all pointing down the axis of the wheel. We will define the total angular momentum of the wheel as the vector sum of the individual angular momentum vectors \( \vec{l}_i \)

\[
\vec{L} \left( \text{total angular momentum of wheel} \right) = \sum_i \vec{l}_i = \sum_i m_i \vec{r}_i \times \vec{v}_i \tag{26}
\]

It is easy to add the vectors \( \vec{l}_i \) because they all point in the same direction, as shown in Figure (15). Thus we can add their magnitudes numerically. (It is just the numerical sum we did back in Equation 15).

**Figure 15**
*Since all the angular momentum vectors \( \vec{l}_i \) point in the same direction, we can add them up numerically.*
To do the sum starting from Equation (26) we note that for each mass $m_i$, the vectors $\vec{r}_i$ and $\vec{v}_i$ are perpendicular, thus

$$\left| \vec{r}_i \times \vec{v}_i \right| = r v \sin \theta = r v \quad \text{(for } \theta = 90^\circ)$$

Then note that for a rotating wheel, the speed $v$ of the rim is related to the angular velocity $\omega$ by

$$v = r \omega \quad \text{(11 repeated)}$$

so that

$$\left| \vec{r}_i \times \vec{v}_i \right| = r v = r (r \omega) = r^2 \omega \quad \text{(27)}$$

Finally note that the vector $\vec{\omega}$ points in the same direction as $\vec{r}_i \times \vec{v}_i$, so that Equation 27 can be written as the vector equation

$$\vec{r}_i \times \vec{v}_i = r^2 \vec{\omega} \quad \text{for all } mass \ m_i \quad \text{(28)}$$

Using Equation 28 in Equation 26 gives

$$\vec{L} = \sum_i \vec{\ell}_i = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

$$= \left( \sum_i m_i \right) r^2 \vec{\omega}$$

$$= M r^2 \vec{\omega}$$

$$\vec{L} = M r^2 \vec{\omega} \quad \text{angular momentum of a rotating bicycle wheel} \quad \text{(29)}$$

where $M$ is the sum of the individual masses $m_i$. Equation 29 is the desired vector version of our original Equation 16.

The important point to get from the above discussion is that by using the vector cross product definition of angular momentum $\vec{\ell}_i = m_i \vec{r}_i \times \vec{v}_i$, all the $\vec{\ell}_i$ for each mass in the wheel pointed down the axis of the wheel, and we could thus calculate the total angular momentum by numerically adding up the individual $\vec{\ell}_i$.

**The $\vec{r} \times \vec{p}$ Definition of Angular Momentum**

A slight rewriting of our definition of angular momentum, Equation 25, gives us a more compact, easily remembered result. Noting that the linear momentum $\vec{p}$ of a particle is $\vec{p} = m \vec{v}$, then a particle’s angular momentum $\vec{\ell}$ can be written

$$\vec{\ell} = m \vec{r} \times \vec{v} = \vec{r} \times (m \vec{v})$$

$$\vec{\ell} = \vec{r} \times \vec{p} \quad \text{(30)}$$

In Chapter 7, we saw that the magnitude of the angular momentum $\vec{\ell}$ of a particle was given by the formula

$$\ell = r_p p \quad \text{(7-15)}$$

where $r_p$ was the lever arm or perpendicular distance from the path of the particle to the point $O$ about which we were measuring the angular momentum. This was illustrated in Figure (7-10) (reproduced here), where a ball of momentum $\vec{p}$, passing by an axis $O$, is caught by a hook and starts rotating in a circle.

![Figure 7-10](image-url)

*As the ball is caught by the hook, its angular momentum, about the point $O$, remains unchanged. It is equal to $(r_p p)$.*
After the ball is caught it is traveling in a circle with an angular momentum \( \mathbf{\ell} = \mathbf{r} \cdot \mathbf{mv} = \mathbf{r} \cdot \mathbf{p} \). By defining the angular momentum as \( \mathbf{r} \cdot \mathbf{p} \) even before the ball was caught, we could say that the ball had the same angular momentum \( \mathbf{r} \cdot \mathbf{p} \) before it was caught by the hook as it did afterward; that the angular momentum was unchanged when the ball was grabbed by the hook.

The idea that the angular momentum is the linear momentum times the perpendicular lever arm \( r_\perp \) follows automatically from the cross product definition of angular momentum \( \mathbf{\ell} = \mathbf{r} \times \mathbf{p} \). To see this, consider a ball with momentum \( \mathbf{p} \) moving past an axis \( O \) as shown in Figure (16a). At the instant of time shown, the ball is located at a coordinate vector \( \mathbf{r} \) from the axis. The angle between the vectors \( \mathbf{r} \) and \( \mathbf{p} \) is the angle \( \theta \) shown in Figure (16b). The vector cross product \( \mathbf{r} \times \mathbf{p} \) is given

\[
\left| \mathbf{\ell} \right| = \left| \mathbf{r} \times \mathbf{p} \right| = rp \sin \theta \tag{31}
\]

However we note that the lever arm or perpendicular distance \( r_\perp \) is given from Figure (16a)

\[
r_\perp = r \sin \theta \tag{32}
\]

Combining Equations 31 and 32 gives

\[
\left| \mathbf{\ell} \right| = \left| \mathbf{r} \times \mathbf{p} \right| = (r \sin \theta)p = r_\perp p \tag{33}
\]

which is the result we used back in Chapter 7.

**Exercise 7**

Using the vectors \( \mathbf{r} \) and \( \mathbf{p} \) in Figure (16), does the vector \( \mathbf{\ell} = \mathbf{r} \times \mathbf{p} \) point up out of the paper or down into the paper?

The intuitive point you should get from this discussion is that the magnitude of the vector cross product \( \mathbf{r} \times \mathbf{p} \) is equal to the magnitude of \( \mathbf{p} \) times the perpendicular lever arm \( r_\perp \). We will shortly encounter the cross product \( \mathbf{r} \times \mathbf{F} \) where \( \mathbf{F} \) is a force vector. We will immediately know that the magnitude of \( \mathbf{r} \times \mathbf{F} \) is \( r_\perp F \) where again \( r_\perp \) is a perpendicular lever arm.

**Figure 16a**
The coordinate vector \( \mathbf{r} \) and the lever arm \( r_\perp \) are related by \( r_\perp = r \sin \theta \).

**Figure 16b**
The angle between \( \mathbf{r} \) and \( \mathbf{p} \) is \( \theta \).
ANGULAR ANALOGY TO NEWTON’S SECOND LAW

We now have the mathematical machinery we need to formulate a complete angular analogy to Newton’s second law. We do this by noting that to go from linear momentum $\mathbf{p}$ to angular momentum $\mathbf{l}$, we took the cross product with the coordinate vector $\mathbf{r}$

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}$$  \hspace{1cm} (30 repeated)

The origin of the coordinate vector $\mathbf{r}$ is the point about which we wish to calculate the angular momentum.

To obtain a dynamical equation for angular momentum $\mathbf{l}$, we start with Newton’s second law which is a dynamical equation for linear momentum $\mathbf{F}$

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$  \hspace{1cm} (11-16)

where $\mathbf{F}$ is the vector sum of the forces acting on the particle.

With one mathematical trick, we can reexpress Newton’s second law in terms of angular momentum. The mathematical trick involves evaluating the expression

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p})$$  \hspace{1cm} (34)

In the ordinary differentiation of the product of two functions $a(t)$ and $b(t)$, we would have

$$\frac{d}{dt}(ab) = \left(\frac{da}{dt}\right)b + a\left(\frac{db}{dt}\right)$$  \hspace{1cm} (35)

The same rules apply if we differentiate a vector cross product. Thus

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \left(\frac{d\mathbf{r}}{dt}\right) \times \mathbf{p} + \mathbf{r} \times \left(\frac{d\mathbf{p}}{dt}\right)$$  \hspace{1cm} (36)

Equation 36 can be simplified by noting that

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

so that

$$\left(\frac{d\mathbf{r}}{dt}\right) \times \mathbf{p} = \mathbf{v} \times \mathbf{p} = \mathbf{v} \times (m\mathbf{v}) = 0$$  \hspace{1cm} (37)

This product is zero because the vectors $\mathbf{v}$ and $\mathbf{p} = m\mathbf{v}$ are parallel to each other, and the cross product of parallel vectors is zero. Thus Equation 36 becomes

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$  \hspace{1cm} (38)

With this result, let us return to Newton’s law for linear momentum

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$  \hspace{1cm} (39)

As long as we do the same thing to both sides of an equation, it is still a correct equation. Taking the vector cross product $\mathbf{r} \times$ on both sides gives

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$  \hspace{1cm} (40)

Using Equation 38 in Equation 40 gives

$$\mathbf{r} \times \mathbf{F} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p})$$  \hspace{1cm} (41)

Finally note that $\mathbf{r} \times \mathbf{p}$ is the particle’s angular momentum $\mathbf{l}$, thus

$$\mathbf{r} \times \mathbf{F} = \frac{d\mathbf{l}}{dt}$$  \hspace{1cm} (42)

Equation (39) told us that the net linear force is equal to the time rate of change of linear momentum. Equation 42 tells us that something, $\mathbf{r} \times \mathbf{F}$, is equal to the time rate of change of angular momentum. What should we call this quantity $\mathbf{r} \times \mathbf{F}$? The obvious name, from an angular analogy would be an angular force. Then we could say that the angular force is the time rate of change of angular momentum, just as the linear force is the time rate of change of linear momentum.

The world does not use the name angular force for $\mathbf{r} \times \mathbf{F}$. Instead it uses the name torque, and usually designates it by the Greek letter $\tau$ (“tau”)

$$\text{torque } \tau = \mathbf{r} \times \mathbf{F}$$  \hspace{1cm} (43)

With this naming, the angular analogy to Newton’s second law is

$$\tau = \frac{d\mathbf{l}}{dt}$$  \hspace{1cm} (44)
ABOUT TORQUE

To gain an intuitive picture of the concept of torque $\tau = \vec{r} \times \vec{F}$, imagine that we have a bicycle wheel with a fixed axis, and push on the rim of the wheel with a force $\vec{F}$ as shown in Figure (17). In (17a) the force $\vec{F}$ is directed through the axis of the wheel, in this case the force has no lever arm $r_\perp$. In (17b), the force is applied above the axis, while in (17c) the force is applied below the axis.

Intuitively, you can see that the wheel will not start turning if you push right toward the axis. When you push above the axis as in (17b), the wheel will start to rotate counter clockwise. By our right hand convention this corresponds to an angular momentum directed up out of the paper.

In (17c), where we push below the axis, the wheel will start to rotate clockwise, giving it an angular momentum directed down into the paper.

FIGURE 17

Both a force $F$ and a lever arm $r_\perp$ are needed to turn the bicycle wheel. The product $r_\perp F$ is the magnitude of the torque $\tau$ acting on the wheel.

Exercise 8

In Figure (17) we have separately drawn the vectors $\vec{F}$ and $\vec{r}$ for each diagram. Using the right hand rule for cross products, find the direction of $\vec{\tau} = \vec{r} \times \vec{F}$ for each of these three diagrams.

If you did Exercise 8 correctly, you found that $\vec{r} \times \vec{F} = 0$ for Figure (17a), that $\vec{\tau} = \vec{r} \times \vec{F}$ pointed up out of the paper in (17b), and down into the paper in (17c). Thus we find that when we apply a zero torque as in (17a), we get zero change in angular momentum. In (17b) we applied an upward directed torque, and saw that the wheel would start to turn to produce an upward directed angular momentum. In (17c), the downward directed torque produces a downward directed angular momentum. These are all results we would expect from the equation $\vec{\tau} = \frac{d\vec{\Omega}}{dt}$.

In our discussion of angular momentum, we saw that $\vec{\tau} = \vec{r} \times \vec{p}$ had a magnitude $|\vec{\tau}| = r_\perp |\vec{p}|$ where $r_\perp$ was the perpendicular lever arm. A similar result applies to torque. By the same mathematics we find that the magnitude of the torque $\vec{\tau}$ produced by a force $\vec{F}$ is

$$|\vec{\tau}| = r_\perp F$$

(45)

where $r_\perp$ is the perpendicular lever arm seen in Figures (17b,c).

Intuitively, the best way to remember torque is to think of it as a force times a lever arm. To turn an object, you need both a force and a lever arm. In Figure (17a), we had a force but no lever arm. The line of action of the force went directly through the axis, with the result that the wheel did not start turning. In both cases (17b) and (17c), there was both a force and a lever arm $r_\perp$, and the wheel started turning.

To get the direction of the torque, to determine whether $\vec{\tau}$ points up or down (and thus gives rise to an up or down angular momentum), use the right hand rule applied to the vector cross product $\vec{\tau} = \vec{r} \times \vec{F}$. A convention, which we will use in the next chapter on Equilibrium, is to say that a torque that points up out of the paper is a positive torque, while a torque pointing down into the paper is a negative one. With this convention, we see that the force in Figure (17b) is exerting a positive torque (and creating positive angular momentum), while the force in Figure (17c) is producing a negative torque (and creating negative angular momentum).
CONSERVATION OF ANGULAR MOMENTUM

In our discussion of a system of particles in Chapter 11, we saw that if we had a system of many interacting particles, with internal forces $F_{i, \text{internal}}$ between the particles, as well as various external forces $F_{i, \text{external}}$, we obtained the equation

$$\sum_i F_{i, \text{external}} = \frac{d\vec{P}}{dt}$$  

(11-12)

where $F_{\text{external}}$ is the vector sum of all the external forces acting on the system, and $\vec{P}$ is the vector sum of all the momenta $\vec{p}_i$ of the individual particles. This result was obtained using Newton’s third law and noting that all the internal forces cancel in pairs. In the case where there is no net external force acting on the system, then $\frac{d\vec{P}}{dt} = 0$ and the total linear momentum of the system is conserved.

We can obtain a similar result for angular momentum by starting with the definition of the total angular momentum $\vec{L}$ of a system as being the vector sum of the angular momentum of the individual particles $\vec{L}_i$:

$$\vec{L} = \sum_i \vec{L}_i$$  

(46)

**definition of the total angular momenta of a system of particles**

Differentiating Equation (46) with respect to time gives

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{L}_i}{dt}$$  

(47)

For an individual particle $i$, we have

$$\frac{d\vec{L}_i}{dt} = \vec{\tau} = \vec{r}_i \times \vec{F}_i$$  

(48)

where $\vec{F}_i$ is the vector sum of the forces acting on the particle $i$. As shown in Figure (18), we can take $\vec{r}_i$ to be the coordinate vector of the $i$-th particle. For this discussion, we can locate the origin of the coordinate system anywhere we want.

Substituting Equation (48) into Equation (47) gives

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i \vec{r}_i \times \vec{F}_i$$

Now break the net force $\vec{F}_i$ into the sum of the external forces $\vec{F}_{i, \text{external}}$ and the sum of the internal forces $\vec{F}_{i, \text{internal}}$. This gives

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_{i, \text{external}} + \sum_i \vec{r}_i \times \vec{F}_{i, \text{internal}}$$  

(49)

Next assume that all the internal forces are equal and opposite as required by Newton’s third law, and are directed toward or away from each other. In Figure (19) we consider a pair of such internal forces and note that both coordinate vectors $\vec{r}_1$ and $\vec{r}_2$ have the same perpendicular lever arm $\vec{r}_\perp$. Thus the equal and opposite forces $\vec{F}_{1, \text{external}}$ and $\vec{F}_{2, \text{internal}}$ create equal and opposite torques which cancel each other in Equation (49). The result is that all torques produced by internal forces cancel in pairs, and we are left with the general result

$$\vec{\tau}_{\text{external}} = \frac{d\vec{L}}{dt}$$  

(50)

where $\vec{\tau}_{\text{external}}$ is the vector sum of all the external torques acting on the system of particles, and $\vec{L}$ is the vector sum of the angular momentum of all of the particles.

---

**Figure 18**  
Coordinate vector for the $i$-th particle.

**Figure 19**  
Both coordinate vectors $\vec{r}_1$ and $\vec{r}_2$ have the same perpendicular lever arm $\vec{r}_\perp$. 
In order to define torque or angular momentum, we have to choose an axis or origin for the coordinate vectors $\vec{r}_i$. (Both torque and angular momentum involve the lever arm $\vec{r}_i$ about that axis.) Equation 50 is remarkably general in that it applies, no matter what origin or axis we choose. In general, choosing a different axis will give us different sums of torques and a different total angular momentum, but the new torques and angular momenta will still obey Equation 50.

In some cases, there is a special axis about which there is no external torque. In the bicycle wheel demonstrations where we stood on a rotating platform, the freely rotating platform did not contribute any external torques about its own axis, which we called the $z$ axis. As long as we did not touch another person or some furniture, then the $z$ component of the external torques was zero. Since Equation 50 is a vector equation, that implies

$$\tau_{z\text{ external}} = \frac{dL_z}{dt} = 0$$

(51)

and we predict that the $z$ component of the total angular momentum (us and the bicycle wheel) should be unchanged, remain constant, no matter how we turned the bicycle wheel. This is just what we saw.

Another consequence of Equation 50 is that if we have an isolated system of particles with no net external torque acting on it, then the total angular momentum will be unchanging, will be conserved. This is one statement of the law of conservation of angular momentum. Our derivation of this result relied on the assumption of Newton's third law that all internal forces are equal and opposite and directed toward each other. Since angular momentum is conserved on an atomic, nuclear and subnuclear scale of distance, where Newtonian mechanics no longer applies, our derivation is in some sense backwards. We should start with the law of conservation of angular momentum as a fundamental law, and show for large objects which obey Newtonian mechanics, the sum of the internal torques must cancel. This is the kind of argument we applied to the conservation of linear momentum in Chapter 11 (see Equation 11-14).

**Figure 7–15 repeated**

*Since the platform is completely free to rotate about the $z$ axis, there are no $z$ directed external torques acting on the system consisting of the platform, person and bicycle wheel. As a result the $z$ component of angular momentum is conserved when the bicycle wheel is turned over.*

*(Note: when the wheel is being held up, we are looking at the under side.)*
GYROSCOPES
The gyroscope provides an excellent demonstration of the predictive power of the equation \( \tau = dL/dt \). Gyroscopes behave in peculiar, non-intuitive ways. The fact that a relatively straightforward application of the equation \( \tau = dL/dt \) predicts this bizarre behavior, provides a graphic demonstration of the applicability of Newton's laws from which the equation is derived.

Start-up
For this discussion, a bicycle wheel with a weighted rim will serve as our example of a gyroscope. To weight the rim, remove the tire and wrap copper wire around the rim to replace the tire. The axle needs to be extended as shown in Figure (20).

As an introduction to the gyroscope problem, start with the bicycle wheel at rest, hold the axle fixed, and apply a force \( F \) to the rim as shown in Figure (20). The force shown will cause the wheel to start spinning in a direction so that the angular momentum \( L \) points to the right as shown. (Curl the fingers of your right hand in the direction of rotation and your thumb points in the direction of \( L \).)

The force \( \vec{F} \), in Figure (20), produces a torque \( \vec{\tau} = \vec{r} \times \vec{F} \) that also points to the right as shown. (The right hand convention used here is to point your fingers in the direction of the first vector \( \vec{r} \), curl them in the direction of the second vector \( \vec{F} \), and your thumb points in the direction of the cross product \( \vec{r} \times \vec{F} = \vec{\tau} \).)

When we start with the bicycle wheel at rest, and apply the right directed torque shown in Figure (20), we get a right directed angular momentum \( \vec{L} \). Thus the torque \( \vec{\tau} \) and the resulting angular momentum \( \vec{L} \) point in the same direction. In addition, the longer we apply the torque, the faster the wheel spins, and the greater the angular momentum \( \vec{L} \). Thus both the direction and magnitude of \( \vec{L} \) are consistent with the equation \( \vec{\tau} = d\vec{L}/dt \).

![Figure 20](image-url)

*Figure 20*
*Spinning up the bicycle wheel. Note that the resulting angular momentum \( L \) points in the same direction as the applied torque \( \tau \).*

![Figure 25 Movie](image-url)

*Figure 25 Movie*
*The gyroscope really works!*
**Precession**

When we apply the equation \( \vec{\tau} = \frac{d\vec{L}}{dt} \) to a gyroscope that is already spinning, and apply the torque in a direction that is not parallel to \( \vec{L} \), the results are not so obvious.

Suppose we get the bicycle wheel spinning rapidly so that it has a big angular momentum vector \( \vec{L} \), and then suspend the bicycle wheel by a rope attached to the end of the axle as shown in Figure (21).

To predict the motion of the spinning wheel, the first step is to analyze all the external forces acting on it. There is the gravitational force \( mg \) which points straight down, and can be considered to be acting at the center of mass of the bicycle wheel, which is the center of the wheel as shown. Then there is the force of the rope which acts along the rope as shown. No other detectable external forces are acting on our system of the spinning wheel.

One thing we know about the force \( \vec{F}_{\text{rope}} \) is that it acts at the point labeled O where the rope is tied to the axle. If we take the sum of the torques acting on the bicycle wheel about the suspension point O, then \( \vec{F}_{\text{rope}} \) has no lever arm about this point and therefore contributes no torque. The only torque about the suspension point O is produced by the gravitational force \( mg \) whose lever arm is \( \vec{r} \), the vector going from point O down the axle to the center of the bicycle wheel as shown in Figure (21).

![Figure 21](image1.png)

**Figure 21**
*Suspend the spinning bicycle wheel by a rope attached to the axle. The gravitational force \( mg \) has a lever arm \( \vec{r} \) about the axis O. This creates a torque \( \vec{\tau} = \vec{r} \times mg \) pointing into the paper.*

The formula for this gravitational torque \( \vec{\tau}_g \) is

\[
\vec{\tau}_g = \vec{r} \times mg
\]

(52)

The new feature of the gyroscope problem, which we have not encountered before, is that the torque \( \vec{\tau} \) does not point in the same direction as the angular momentum \( \vec{L} \) of the bicycle wheel. If we look at Figure (21), point the fingers of our right hand in the direction of the vector \( \vec{r} \), and curl our fingers in the direction of the vector \( mg \), then our thumb points down into the paper. This is the definition of the direction of the vector cross product \( \vec{r} \times mg \). But the angular momentum \( \vec{L} \) of the bicycle wheel points along the axis of the wheel to the right in the plane of the paper. In order to view both the angular momentum vector \( \vec{L} \) and the torque vector \( \vec{\tau} \) in the same diagram, we can look down on the bicycle wheel from the ceiling as shown in Figure (22).

![Figure 22](image2.png)

**Figure 22**
*Looking down from the ceiling, the vector \( mg \) points down into the paper and \( \vec{\tau} = \vec{r} \times mg \) points to the top of the page. In this view we can see both the vectors \( \vec{L} \) and \( \vec{\tau} \).*

When we started the wheel spinning, back in Figure (20), the torque \( \vec{\tau} \) and angular momentum \( \vec{L} \) pointed in the same direction, and we had the simple result that the longer we applied the torque, the more angular momentum we got. Now, with the torque and angular momentum pointing in different directions as shown in Figure (22), we expect that the torque will cause a change in the direction of the angular momentum.
To predict the change in $L$, we start with the angular form of Newton’s second law

$$\ddot{r} = \frac{dL}{dt}$$

and multiply through by the short (but finite) time interval $dt$ to get

$$dL = \tau dt$$

Equation 53 gives us $dL$, which is the change in the bicycle wheel’s angular momentum as a result of applying the torque $\tau$ for a short time $dt$.

To see the effect of this change $dL$, we will use some of the terminology we used in the computer prediction of motion. Let us call $L_{\text{old}}$ the old value of the angular momentum that the bicycle wheel had before the time interval $dt$, and $L_{\text{new}}$ the new value at the end of the time interval $dt$. Then $L_{\text{new}}$ will be related to $L_{\text{old}}$ by the equation

$$L_{\text{new}} = L_{\text{old}} + dL$$

Using Equation 53 for $dL$ gives

$$L_{\text{new}} = L_{\text{old}} + \tau dt$$

A graph of the vectors $L_{\text{old}}$, $L_{\text{new}}$, and $\tau dt$ is shown in Figure (23). In this figure the perspective is looking down on the bicycle wheel, as in Figure (22).

Since the torque $\tau$ is in the horizontal plane, the vector $L_{\text{new}} = L_{\text{old}} + \tau dt$ is also in the horizontal plane. And since $\tau$ and $L_{\text{old}}$ are perpendicular to each other, $L_{\text{new}}$ has essentially the same length as $L_{\text{old}}$. What is happening is that the vector $L$ is starting to rotate counter clockwise (as seen from above) in the horizontal plane.

One final, important point. For this experiment we were careful to spin up the bicycle wheel so that before we suspended the wheel from the rope, the wheel had a big angular momentum pointing along its axis of rotation. When we apply a torque to change the direction of $L$, the axis of the wheel and the angular momentum vector $L$ move together. As a result the axis of the bicycle wheel also starts to rotate counter clockwise in the horizontal plane. The bicycle wheel, instead of falling as expected, starts to rotate sideways.

Once the bicycle wheel has turned an angle $d\theta$ sideways, the axis of rotation and the torque $\tau$ also rotate by an angle $d\theta$, so that the torque $\tau$ is still perpendicular to $L$ as shown in Figure (24). Since $\tau$ always remains perpendicular to $L$, the vector $\tau dt$ cannot change the length of $L$. Thus the angular momentum vector $L$ remains constant in magnitude and rotates or “precesses” in the horizontal plane. This is the famous precession of a gyroscope which is nicely demonstrated using the bicycle wheel apparatus of Figure (21).
To calculate the rate of precession we note from Figures (23) or (24) that the angle $d\theta$ is given by

$$d\theta = \frac{\tau dt}{L}$$

(56)

where we use the fact that $\tau dt$ is a very short length, and thus $\sin(d\theta)$ and $d\theta$ are equivalent. Dividing both sides of Equation 56 through by $dt$, we get

$$\frac{d\theta}{dt} = \frac{\tau}{L}$$

(57)

But $d\theta/dt$ is just the angular velocity of precession, measured in radians per second. Calling this precessional velocity $\Omega_{\text{precession}}$ ($\Omega$ is just a capital $\omega$ “omega”), we get

$$\Omega_{\text{precession}} = \frac{\tau}{L}$$

(58)

Exercise 9

A bicycle wheel of mass $m$, radius $R$, is spun up to an angular velocity $\omega$. It is then suspended on an axle of length $r$ as shown in Figure (21). Calculate

(a) the angular momentum $L$ of the bicycle wheel.

(b) the angular velocity of precession.

(c) the time it takes the wheel to precess around once (the period of precession). [You should be able to obtain the period of precession from the angular velocity of precession by dimensional analysis.]

(d) a bicycle wheel of total mass 1kg and radius 40cm, is spun up to a frequency $f = 2\pi \omega = 10$ cycles/sec. The handle is 30cm long. What is the period of precession in seconds? Does the result depend on the mass of the bicycle wheel?

If you try the bicycle wheel demonstration that we discussed, the results come out close to the prediction. Instead of falling as one might expect, the wheel precesses horizontally as predicted. There is a slight drop when you let go of the wheel, which can be compensated for by releasing the wheel at a slight upward angle.

If you look at the motion of the wheel carefully, or study the motion of other gyroscopes (particularly the air bearing gyroscope often used in physics lectures) you will observe that the axis of the wheel bobs up and down slightly as it goes around. This bobbing, or epicycle like motion, is called nutation. We did not predict this nutation because we made the approximation that the axis of the wheel exactly follows the angular momentum vector. This approximation is very good if the gyroscope is spinning rapidly but not very good if $L$ is small. Suppose, for example we release the wheel without spinning it. Then it simply falls. It starts to rotate, but along a different axis. As it starts to fall it gains angular momentum in the direction of $\tau$. A more accurate analysis of the motion of the gyroscope can become fairly complex. But as long as the gyroscope is spinning fast enough so that the axis moves with $L$, we get the simple and important results discussed above.
APPENDIX

Moment of Inertia and Rotational Kinetic Energy

In the main part of the text, we briefly discussed moment of inertia as the angular analogy to mass in the formula for angular momentum. As linear momentum $p$ of an object is its mass $m$ times its linear velocity $v$

$$ p = m v \quad \text{linear momentum} \quad (A1) $$

the angular momentum $\vec{I}$ is the angular mass or moment of inertia $I$ times the angular velocity $\omega$

$$ I = I \omega \quad \text{angular momentum} \quad (A2) $$

In the simple case of a bicycle wheel, where all the mass is essentially out at a distance ($r$) from the axis of the wheel, the moment of inertia about the axis is

$$ I = Mr^2 \quad \text{moment of inertia of a bicycle wheel} \quad (A3) $$

where $M$ is the mass of the wheel.

When the mass of an object is not all concentrated out at a single distance ($r$) from the axis, then we have to calculate the moment of inertia of individual parts of the object that are at different radii $r$, and tie together the various pieces to get the total moment of inertia. This usually involves an integration, like the one we did in Equations 21 through 23 to calculate the moment of inertia of a solid cylinder.

For topics to be discussed later in the text, the earlier discussion of moment of inertia is all we need. But there are topics, such as rotational kinetic energy and its connection to moment of inertia, which are both interesting, and can be easily tested in both lecture demonstrations and laboratory exercises. We will discuss these topics here.

**ROTATIONAL KINETIC ENERGY**

Let us go back to our example, shown in Figure (3) repeated here, of a ball of mass $m$, on the end of a massless stick of length $r$, rotating with an angular velocity $\omega$. The speed $v$ of the ball is given by Equation 11 as

$$ v = r \omega \quad (11 \text{ repeated}) $$

and the ball’s kinetic energy will be

$$ \text{kinetic energy} = \frac{1}{2} mv^2 = \frac{1}{2} m (r \omega)^2 $$

$$ = \frac{1}{2} (mr^2) \omega^2 \quad (A4) $$

Since the ball’s moment of inertia $I$ about the axis of rotation is $mr^2$, we get as the formula for the ball’s kinetic energy

$$ \text{kinetic energy} = \frac{1}{2} I \omega^2 \quad \text{analogous to } \frac{1}{2} mv^2 \quad (A5) $$

We see the angular analogy working again. The ball has a kinetic energy, due to its rotation, which is analogous to $\frac{1}{2} mv^2$, with the linear mass $m$ replaced by the angular mass $I$ and the linear velocity $v$ replaced by the angular velocity $\omega$.

**Figure 3 repeated**

*Mass rotating on the end of a massless stick.*
If we have a bicycle wheel of mass $M$ and radius $r$ rotating at an angular velocity $\omega$, we can think of the wheel as being made up of a collection of masses on the ends of rods as shown in Figure (6) repeated here. For each individual mass $m_i$, the kinetic energy is $\frac{1}{2} m_i v^2$ where $v = r \omega$ is the same for all the masses. Thus the total kinetic energy is

$$\text{kinetic energy of bicycle wheel} = \sum_i \frac{1}{2} m_i r^2 \omega^2$$

$$= \frac{1}{2} r^2 \omega^2 \sum_i m_i$$

$$= \frac{1}{2} r^2 \omega^2 M$$

where the sum of the masses $\sum m_i$ is just the mass $M$ of the wheel. The result can now be written

$$\text{kinetic energy of bicycle wheel} = \frac{1}{2} (Mr^2) \omega^2$$

$$= \frac{1}{2} I \omega^2 \quad \text{(A6)}$$

If we call $Mr^2$ the angular mass, or moment of inertia $I$ of the bicycle wheel, we again get the formula $1/2 I \omega^2$ for kinetic energy of the wheel. Thus we see that, in calculating this angular mass or moment of inertia, it does not make any difference whether the mass is concentrated at one point as in Figure (3), or spread out as in Figure (6). The only criterion is that the mass or masses all be out at the same distance $r$ from the axis of rotation.

In most of our examples we will consider objects like bicycle wheels or hollow cylinders where the mass is essentially all at a distance $r$ from the axis of rotation, and we can use the formula $Mr^2$ for the moment of inertia. But often the mass is spread out over different radii and we have to calculate the angular mass. An example is a rotating shaft shown back in Figure (9), where the mass extends from the center where $r = 0$ out to the outside radius $r = R$.

Suppose we have an arbitrarily shaped object rotating an angular velocity $\omega$ about some axis, as shown in Figure (A1). To find the moment of inertia, we will calculate the kinetic energy of rotation and equate that to $1/2 I \omega^2$ to obtain the formula for $I$. To do this we conceptually break the object into many small masses $dm_i$ located a distance $r_i$ from the axis of rotation as shown. Each $dm_i$ will have a speed $v_i = r_i \omega$, and thus a kinetic energy

$$\text{kinetic energy of object} = \sum_i \left( \frac{1}{2} m_i v_i^2 \right)$$

$$= \sum_i \left( \frac{1}{2} m_i r_i^2 \omega^2 \right)$$

$$= \frac{1}{2} \omega^2 \sum_i \left( m_i r_i^2 \right)$$

$$= \frac{1}{2} \omega^2 I \quad \text{(A7)}$$

From Equation A7, we see that the general formula for moment of inertia is

$$I = \sum_i m_i r_i^2 \quad \text{(A8)}$$
In Example 1, Equations 21 through 23, we showed you how to calculate the moment of inertia of a solid cylinder about its axis of symmetry. In that example we broke the cylinder up into a series of concentric shells of radius \( r_i \) and mass \( dm_i \), calculated the moment of inertia of each shell \( dm_i r_i^2 \), and summed the results as required by Equation A7. As in most cases where we calculate a moment of inertia, the sum is turned into an integral.

In Exercise 3 which followed Example 1, we had you calculate the moment of inertia, about its axis of symmetry, of a hollow thick-walled cylinder. The calculation was essentially the same as the one we did in Example 1, except that you had to change the limits of integration. The following exercise gives you more practice calculating moments of inertia, and shows you what happens when you change the axis about which the moment of inertia is calculated.

**Exercise A1**

Consider a uniform rod of mass \( M \) and length \( L \) as shown in Figure (A2).

a) calculate the moment of inertia of the rod about the center axis, labeled axis 1 in Figure (A2).

b) calculate the moment of inertia of the rod about an axis that goes through the end of the rod, axis 2 in Figure (A2). About which axis is the moment of inertia greater? Explain why.

![Figure A2](image)

Calculating the moment of inertia of a long thin rod.

**COMBINED TRANSLATION AND ROTATION**

In our discussion of the motion of a system of particles, we saw that the motion was much easier to understand if we focused our attention on the motion of the center of mass of the system. The simple feature of the motion of the center of mass, was that the effects of all internal forces cancelled. The center of mass moved as if it were a point particle of mass \( M \), equal to the total mass of the system, subject to a force \( F \) equal to the vector sum of all the external forces acting on the object.

When the system is a rigid object, we have a further simplification. The motion can then be described as the motion of the center of mass, plus rotation about the center of mass. To see that you can do this, imagine that you go to a coordinate system that moves with the object’s center of mass. In that coordinate system, the object’s center of mass point is at rest, and the only thing a rigid solid object can do is rotate about that point.

A key advantage of viewing the motion of a rigid object this way is that the kinetic energy of a moving, rotating, solid object is simply the kinetic energy of the center of mass motion plus the kinetic energy of rotation. Explicitly, if an object has a total mass \( M \), and a moment of inertia \( I_{\text{com}} \) about the center of mass (parallel to the axis of rotation of the object) then the formula for the kinetic energy of the object is

\[
\text{kinetic energy of moving and rotating object} = \frac{1}{2} M V_{\text{com}}^2 + \frac{1}{2} I_{\text{com}} \omega^2 \quad (A9)
\]

where \( V_{\text{com}} \) is the velocity of the center of mass and \( \omega \) the angular velocity of rotation about the center of mass.

More important is the idea that motion can be separated into the motion of the center of mass plus rotation about the center of mass. To emphasize the usefulness of this concept, we will first consider an example that can easily be studied in the laboratory or at home, and then go through the proof of the equation.
Example—Objects Rolling Down an Inclined Plane

Suppose we start with a cylindrical object at the top of an inclined plane as shown in Figure (A3), and measure the time the cylinder takes to roll down the plane. Since we do not have to worry about friction for a rolling object, we can use conservation of energy to analyze the motion.

If the cylinder rolls down so that its height decreases by \( h \) as shown, then the loss of gravitational potential energy is \( mgh \). Equating this to the kinetic energy gained gives

\[
mgh = \frac{1}{2} m v_{\text{com}}^2 + \frac{1}{2} I \omega^2 \tag{A10}
\]

where \( m \) is the mass of the cylinder, \( v_{\text{com}} \) the speed of the axis of the cylinder, \( I \) the moment of inertia about the axis and \( \omega \) the angular velocity.

If the cylinder rolls without slipping, there is a simple relationship between \( v_{\text{com}} \) and \( \omega \). We are picturing the rolling cylinder as having two kinds of motion—translation and rotation. The velocity of any part of the cylinder is the vector sum of \( v_{\text{com}} \) plus the velocity due to rotation.

At the point where the cylinder touches the inclined plane, the rotational velocity has a magnitude \( v_{\text{rot}} = \omega r \), and is directed back up the plane as shown in Figure (15). If the cylinder is rolling without slipping, the velocity of the cylinder at the point of contact must be zero, thus we have

\[
v_{\text{com}} + \omega r = 0 \quad \text{rolling without slipping} \tag{A11}
\]

Thus we get for magnitudes

\[
\omega r = v_{\text{com}} ; \quad \omega = \frac{v_{\text{com}}}{r}
\]

Using Equation A12 in A10 gives

\[
mgh = \frac{1}{2} m v_{\text{com}}^2 + \frac{1}{2} I \left( \frac{v_{\text{com}}^2}{r^2} \right) = \frac{1}{2} \left( m + \frac{I}{r^2} \right) v_{\text{com}}^2 \tag{A13}
\]

Let us take a look at what is happening physically as the cylinder rolls down the plane. In our earlier analysis of a block sliding without friction down the plane, all the gravitational potential energy \( mgh \) went into kinetic energy \( 1/2 m v_{\text{com}}^2 \). Now for a rolling object, the gravitational potential has to be shared between the kinetic energy of translation \( 1/2 m v_{\text{com}}^2 \) and the kinetic energy of rotation \( 1/2 I \omega^2 \). The greater the moment of inertia \( I \), the more energy that goes into rotation, the less available for translation, and the slower the object rolls down the plane.

In our discussion of moments of inertia, we saw that for two cylinders of equal mass, the hollow thin-walled cylinder had twice the moment of inertia as the solid one. Thus if you roll a hollow and a solid cylinder down the plane, the solid cylinder will travel faster because less gravitational potential energy goes into the kinetic energy of rotation. You get to figure out how much faster in Exercise A2.

---

**Figure A3**
Calculating the speed of an object rolling down a plane.

**Figure A4**
The velocity at the point of contact is the sum of the center of mass velocity and the rotational velocity. This sum must be zero if there is no slipping.
Before you work Exercise A2, think about this question. The technician who sets up our lecture demonstrations has a metal sphere, and does not know for sure whether the sphere is solid or hollow. (It could be a solid sphere made of a light metal, or a hollow sphere made from a more dense metal.) How could you find out if the sphere is solid or hollow?

Exercise A2
You roll various objects down the inclined plane shown in Figure (A3).

(a) a thin walled hollow cylinder
(b) a solid cylinder
(c) a thin walled sphere
(d) a solid sphere

and for comparison, you also slide a frictionless block down the plane:
(e) a frictionless block

For each of these, calculate the speed \( v_{\text{com}} \) after the object has descended a distance \( h \). (It is easy to do all cases of this problem by writing the object’s moment of inertia in the form \( I = \alpha MR^2 \), where \( \alpha = 1 \) for the hollow cylinder, \( 1/2 \) for the solid cylinder, etc.) What value of \( \alpha \) should you use for the sliding block?

Writing your results in the form \( v_{\text{com}} = \beta \sqrt{2gh} \) summarize your results in a table giving the value of \( \beta \) in each case. (\( \beta = 1 \) for the sliding block, and is less than 1 for all other examples.)

Exercise A3 A Potential Lab Experiment
In Exercise A2 you calculated the speed \( v_{\text{com}} \) of various objects after they had descended a distance \( h \). A block sliding without friction has a speed \( v \) given by \( mgh = 1/2mv^2 \), or \( v = \sqrt{2gh} \). The rolling objects were moving slower when they got to the bottom. For all heights, however, the speed of a rolling object is slower than the speed of the sliding block by the same constant factor. Thus the rolling objects moved down the plane with constant acceleration, but less acceleration than the sliding block. It is as if the acceleration due to gravity were reduced from the usual value \( g \). Using this idea, and the results of Exercise A2, predict how long each of the rolling objects takes to travel down the plane. This prediction can be tested with a stop watch.

\[
\vec{R}_i = \vec{R}_{\text{com}} + \vec{r}_i
\]  (A14)

We can obtain an equation for the velocity of the small mass \( m_i \) by differentiating Equation A14 with respect to time

\[
\frac{d\vec{R}_i}{dt} = \frac{d\vec{R}_{\text{com}}}{dt} + \frac{d\vec{r}_i}{dt}
\]  (A15)

\[\text{Figure A5}
\text{Analyzing the motion of a small piece of an object.}\]
which can be written in the form

\[ \mathbf{v}_i = \mathbf{v}_{\text{com}} + \mathbf{v}_i \]  \hspace{1cm} (A16)

where \( \mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} \) is the velocity of \( m_i \) in our coordinate system, \( \mathbf{v}_{\text{com}} = \frac{d\mathbf{r}_{\text{com}}}{dt} \) is the velocity of the center of mass of the object, and \( \mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} \) is the velocity of \( m_i \) in a coordinate system that is moving with the center of mass of the object.

The kinetic energy of the small mass \( m_i \) is

\[ \frac{1}{2} m_i \mathbf{v}_i^2 = \frac{1}{2} m_i (\mathbf{v}_{\text{com}} + \mathbf{v}_i) \cdot (\mathbf{v}_{\text{com}} + \mathbf{v}_i) \]

\[ = \frac{1}{2} m_i (\mathbf{v}_{\text{com}}^2 + 2 \mathbf{v}_{\text{com}} \cdot \mathbf{v}_i + \mathbf{v}_i^2) \]

\[ = \frac{1}{2} m_i \mathbf{v}_{\text{com}}^2 + \frac{1}{2} m_i \mathbf{v}_i^2 + m_i \mathbf{v}_{\text{com}} \cdot \mathbf{v}_i \]  \hspace{1cm} (A17)

The total kinetic energy of the object is the sum of the kinetic energy of all the small pieces \( m_i \)

\[ \text{total kinetic energy} = \sum_i \frac{1}{2} m_i \mathbf{v}_i^2 \]

\[ = \frac{1}{2} \mathbf{v}_{\text{com}}^2 \sum_i m_i + \sum_i \frac{1}{2} m_i \mathbf{v}_i^2 + \mathbf{v}_{\text{com}} \cdot \sum_i m_i \mathbf{v}_i \]  \hspace{1cm} (A18)

In two of the terms, we could take the common factor \( \mathbf{v}_{\text{com}} \) outside the sum.

Now the quantity \( m_i \mathbf{v}_i \) that appears in the last term of Equation A18 is the linear momentum of \( m_i \) as seen in a coordinate system where the center of mass is at rest.

To evaluate the sum of these terms, let us choose a new coordinate system whose origin is at the center of mass of the object as shown in Figure (A6). In this coordinate system the formula for the center of mass of the small masses \( m_i \) is

\[ \mathbf{r}_{\text{com}} = \sum_i m_i \mathbf{r}_i = 0 \]  \hspace{1cm} (A19)

Differentiating Equation 19 with respect to time gives

\[ \sum_i m_i \frac{d\mathbf{r}_i}{dt} = \sum_i m_i \mathbf{v}_i = 0 \]  \hspace{1cm} (A20)

Equation A20 tells us that when we are moving along with the center of mass of a system of particles, the total linear momentum of the system, the sum of all the \( m_i \mathbf{v}_i \), is zero.

Using Equation A20 in A18 gets rid of the last term. If we let \( M = \sum_i m_i \) be the total mass of the object, we get

\[ \text{total kinetic energy} = \frac{1}{2} M \mathbf{v}_{\text{com}}^2 + \sum_i \frac{1}{2} m_i \mathbf{v}_i^2 \]  \hspace{1cm} (A21)

Equation A21 applies to any system of particles, whether the particles make up a rigid object or not. The first term, \( \frac{1}{2} M \mathbf{v}_{\text{com}}^2 \) is the kinetic energy of center of mass motion, and \( \sum_i \frac{1}{2} m_i \mathbf{v}_i^2 \) is the kinetic energy as seen by someone moving along with the center of mass. If the object is solid, then in a coordinate system where the center of mass is at rest, the only thing the object can do is rotate about the center of mass. As a result the kinetic energy in that coordinate system is the kinetic energy of rotation. If the moment of inertia about the axis of rotation is \( I_{\text{com}} \), then the total kinetic energy is \( \frac{1}{2} M \mathbf{v}_{\text{com}}^2 + \frac{1}{2} I_{\text{com}} \omega^2 \) where \( \omega \) is the angular velocity of rotation. This is the result we stated in Equation A9.

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**Figure A6**

*Here we moved the origin of the coordinate system to the center of mass.*