Chapter 8 non calculus

Newtonian Mechanics

In the special cases of motion with constant acceleration, and uniform circular motion, we had formulas for predicting motion far into the future. Then in our Essay on Predicting Motion, which followed Chapter 5, we discussed a general scheme for predicting the motion of an object whenever we had a formula for the object's acceleration vector.

What was missing in these earlier discussions was a way to find the formula for an object's acceleration vector. To do this, two new concepts are needed. One is ***mass****, discussed in Chapter 6, and the other is* ***force****, to be introduced now. We will see that once we know the forces acting on an object, we can obtain a formula for the object's acceleration and then use our earlier techniques to predict motion. This scheme was developed in the late 1600s by Isaac Newton and is known as* ***Newtonian Mechanics****.*

FORCE

The concept of a force—a push or a pull—is not as strange or unfamiliar as the acceleration vector we have been discussing. When you push on an object you are exerting a force on that object. The harder you push, the stronger the force. And the direction you push is the direction of the force. From this we see that force is a quantity that has a magnitude and a direction. As a result, it is reasonable to assume that a force is described mathematically by a vector, which we will usually designate by the letter $\vec{F}$.

It is often easy to see when forces are acting on an object. What is more subtle is the relationship between force and the resulting acceleration it produces. If I push on a big tree, nothing happens. I can push as hard as I want and the tree does not move. (No bulldozers allowed.) But if I push on a chair, the chair may move. The chair moves if I push sideways but not if I push straight down.

The ancient Greeks, Aristotle in particular, thought that there was a direct relationship between force and velocity. Aristotle thought that the harder you pushed on an object, the faster it went. There is some truth to this, if you are talking about pushing a stone along the ground or pulling a boat through water. But these examples, which were familiar problems in ancient times, turn out to be complex situations, involving friction and viscous forces.

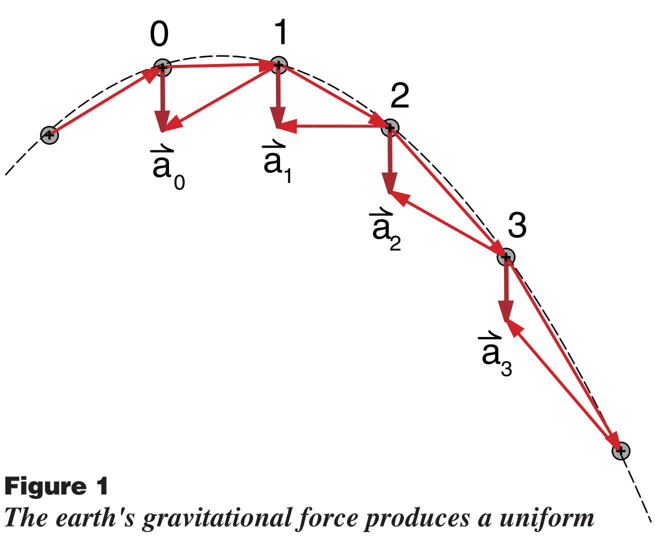


Figure 1
The earth's gravitational force produces a uniform downward gravitational acceleration. (See Figure 3-17)

Only when Galileo focused on a problem without much friction—projectile motion—did the important role of the acceleration vector become apparent. Later, Newton compared the motion of a projectile (the apple that supposedly fell on his head) with the motion of the planets and the moon, giving him more examples of motion without friction. These examples led Newton to the discovery that force is directly related to acceleration, not velocity.

In our discussion of projectile motion, and projectile motion with air resistance, we have begun to see the relation between force and acceleration. While a projectile is in flight, and we can neglect air resistance, the projectile's acceleration is straight down, in the direction of the earth as shown in Figure (1). As we stand on the earth, we are being pulled down by gravity. While the projectile is in flight, it is also being pulled down by gravity. It is a reasonable guess that the projectile's downward acceleration vector $\vec{g}$ is caused by the gravitational force of the earth.

When we considered the motion of a particle at constant speed in a circle as shown in Figure (2), we saw that the particle's acceleration vector pointed toward the center of the circle. A simple physical example of this circular motion was demonstrated when we tied a golf ball to a string and swung it over our head.

While swinging the golf ball, it was the string pulling in on the ball that kept the ball moving in a circle. (Let go of the string and the ball goes flying off.) The string is capable of pulling only along the length of the string, which in this case is toward the center of the circle. Thus the force exerted by the string is in the direction of the golf ball's acceleration vector. This is our second example in which the particle's acceleration vector points in the same direction as the force exerted on it.

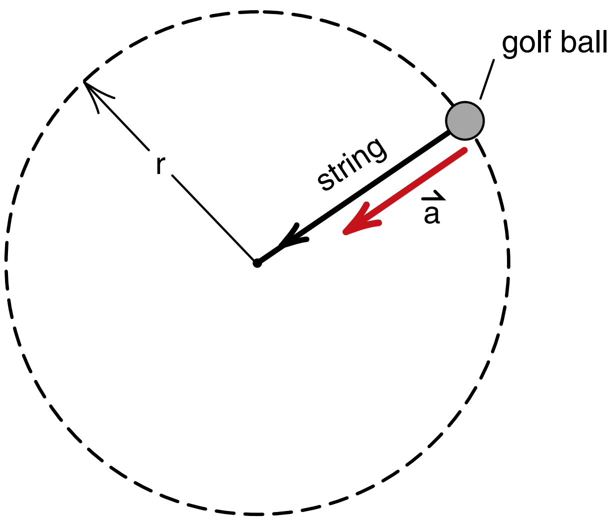


Figure 2
The acceleration of the ball is in the same direction as the force exerted by the string. (See Figure 3-23)

THE ROLE OF MASS

In our two examples, projectile motion and motion in a circle, the force produces an acceleration in the direction of the force. The next question is—how much acceleration? Clearly not all forces have the same effect. If I shove a child's toy wagon, the wagon might accelerate rapidly and go flying off. The same shove applied to a Buick automobile will not do very much.

There is clearly a difference between a toy wagon and a Buick. The Buick has much more mass than the wagon, and is much less responsive to my shove.

In our recoil definition of mass discussed in Chapter 6 and illustrated in Figure (3), we defined the ratio of two masses as the inverse ratio of their recoil speeds

$$\frac{m_a}{m_b} = \frac{v_b}{v_a} \qquad (6\text{-}1)$$

The intuitive idea is that the more massive the object, the slower it recoils. The more mass, the less responsive it is to the shove that pushed the carts apart.

Think about the spring that pushes the carts apart in our recoil experiment. Once we burn the thread holding the carts together, the spring pushes out on both carts, causing them to accelerate outward. If the spring is pushing equally hard on both carts (later we will see that it must), then we see that the resulting acceleration and final velocities are inversely proportional to the mass of the cart. If m_b is twice as massive as m_a , it gets only half as much acceleration from the same spring force. Our recoil definition and experiments on mass suggest that the effectiveness of a force in producing an acceleration is inversely proportional to the object's mass. For a given force, if you double the mass, you get only half the acceleration. That is the simplest relationship between force and mass that is consistent with our general experience, and it turns out to be the correct one.

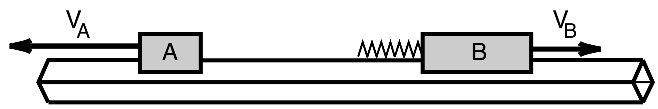


Figure 3
Definition of mass. When two carts recoil from rest, the more massive cart recoils more slowly. (See Figure 6-3)

NEWTON'S SECOND LAW

We have seen that a force $\vec{F}$ acting on a mass m, produces an acceleration $\vec{a}$ that 1) is in the direction of $\vec{F}$, and 2) has a magnitude inversely proportional to m. The simplest equation consistent with these observations is

$$\vec{a} = \frac{\vec{F}}{m} \tag{1}$$

Equation (1) turns out to be the correct relationship, and is known as ***Newton's Second Law of Mechanics***. (The ***First Law*** is a statement of the special case that, if there are no forces, there is no acceleration. That was not obvious in the late 1600s, and was therefore stated as a separate law.) A more familiar form of Newton's second law, seen in all introductory physics texts, is

$$\boxed{\vec{F} = m\vec{a}} \tag{1a}$$

If there is any equation that is essentially an icon for the introductory physics course, Equation (1a) is it.

At this point, Equation (1) or (1a) serves more as a definition of force than a basic scientific result. We can, for example, see from Equation (1a) that force has the dimensions of mass times acceleration. In the MKS system of units this turns out to be kg(m/sec^2), a collection of units called the ***newton***. Thus we can say that we push on an object with a force of so many newtons.

In the CGS system, the dimensions of force are gm(cm/sec^2), a set of units called a ***dyne***. A dyne turns out to be a very small unit of force, of the order of the force exerted by a fly doing push-ups. The only time we will use dynes is in our study of atomic physics where the CGS system is more intuitive than the MKS system.

The real confusion is in the English system of units where force is measured in *pounds*, and the unit of mass is a *slug*. We will avoid doing Newton's law calculations in English units so that the student does not have to worry about pounds and slugs.

Detecting Forces

From another point of view, we can use Newton's second law, Equation (1), to ***detect the existence of a force by the acceleration it produces***. In our first strobe photograph of a steel ball projectile, we saw that the ball's acceleration vector $\vec{g}$ pointed straight down and had a magnitude of about 9.8 meters/sec^2. If we say that this acceleration is produced by a gravitational force $\vec{F}_g$ acting on the ball, then Equation (1a) with $\vec{a} = \vec{g}$ gives us the formula for the gravitational force

$$\boxed{\vec{F}_g = m\vec{g}} \quad \textit{gravitational force on a mass m} \tag{2}$$

where m is the mass of the ball.

Several Forces

We have been applying Newton's second law to special cases where only one force is acting on an object, and we can directly see what acceleration that force produces. These are rare examples. Usually more than one force acts at the same time, and the acceleration we observe is produced by a combination of the forces. How the forces combine is fairly well illustrated by our analysis of the Styrofoam ball projectile where air resistance noticeably affected the motion of the ball.

To understand where the air resistance force came from, we had you imagine that you were riding with the ball and felt a wind in your face due to the motion as shown in Figure (4). The wind would exert a force on you and the ball in much the same way you feel the force of the wind on your hand when you stick your hand out of a moving car window.

In Figure (4) we see that the observed acceleration $\vec{a}_{ball}$ of the ball is the vector sum of the acceleration $\vec{g}$ due to gravity and a vector $\vec{a}_{air}$ that points in the direction of the wind in your face. We have the vector equation

$$\vec{a}_{ball} = \vec{g} + \vec{a}_{air} \tag{3}$$

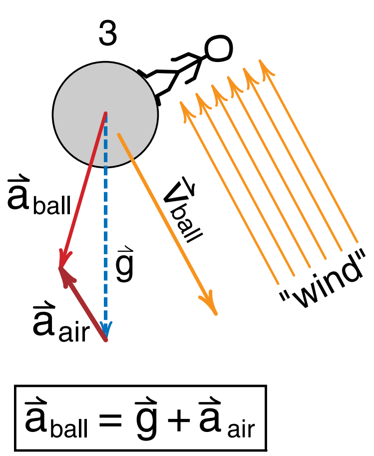


Figure 4
Gravity and the wind each produce an acceleration, $\vec{g}$ and $\vec{a}_{air}$ respectively. The net acceleration of the ball is the vector sum of the two accelerations.

Multiplying Equation (3) through by the ball's mass m we get

$$m\vec{a}_{ball} = m\vec{g} + m\vec{a}_{air} \tag{4}$$

If we say that the air resistance acceleration $\vec{a}_{air}$ is caused by an air resistance force $\vec{F}_{air} = m\vec{a}_{air}$, and note that $m\vec{g} = \vec{F}_g$ is the gravitational force on the ball, we get

$$\boxed{m\vec{a}_{ball} = \vec{F}_g + \vec{F}_{air}} \tag{5}$$

Equation (5) provides an example of how to use Newton's second law when more than one force is acting on an object. First find the vector sum of the forces acting, in this case $(\vec{F}_g + \vec{F}_{air})$, and then equate that to the object's mass m times its observed acceleration $\vec{a}$.

Another way to write Newton's second law that reminds us to first take the vector sum of the forces is

$$\boxed{\vec{F}_{total} = m\vec{a}} \quad \textit{more general form of Newton's second law} \tag{6}$$

where $\vec{F}_{total}$ is the vector sum of all the forces acting on an object of mass m, and $\vec{a}$ is the object's acceleration.

ZERO ACCELERATION

What if an object is not accelerating? For example, suppose I am standing still holding the suitcase shown in Figure (5). What can I say about the forces acting on the suitcase? If the suitcase acceleration vector is zero, then Equation (6) gives

$$\vec{F}_{total} = 0 \tag{7}$$

where $\vec{F}_{total}$ is the vector sum of the forces acting on the suitcase.

As I stand there, I am fully aware that I have to pull up on the suitcase to keep it from falling to the ground. Let us call the upward force I exert $\vec{F}_{me}$. In addition we also know that gravity is pulling down with a force $\vec{F}_g = m\vec{g}$. If nothing else is pushing or pulling on the suitcase, then we have

$$(F_{total})_{on\ suitcase} = \vec{F}_{me} + \vec{F}_g = 0 \tag{8}$$

Solving this fairly simple equation gives

$$\vec{F}_{me} = -\vec{F}_g = -m\vec{g} \tag{9}$$

In other words I have to pull up (opposite to $\vec{g}$) with a force of magnitude mg.

If I set the suitcase down on the floor, and leave it there, the suitcase still has no acceleration and the total force must still be zero. Now the force I exerted to hold up the suitcase is replaced by the upward force $\vec{F}_{floor}$ that the floor exerts on the suitcase. Thus the floor must now be pushing upward with a force of magnitude mg as shown in Figure (6).

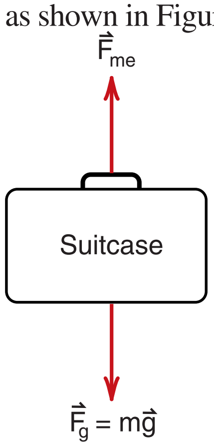


Figure 5
The two forces acting on the suitcase while I am holding it.

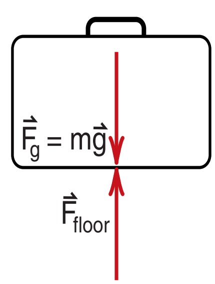


Figure 6
The two forces acting on the suitcase while it is sitting on the floor.

NEWTON'S THIRD LAW

When Newton introduced his laws of mechanics, he presented them as three separate laws. As we mentioned, the first was that if no net force is acting on an object, the object will not accelerate, and the second law is the more general statement that a net force causes an acceleration according to Equation (1). The third law is that ***when two bodies interact, the forces they exert on each other are always equal in magnitude and oppositely directed***.

As a simple example of this law, consider the case we just discussed where we set a suitcase down on the floor. We saw that the floor had to push up on the suitcase with a force $\vec{F}_{floor} = -m\vec{g}$ in order to prevent the suitcase from accelerating. Newton's third law requires that the suitcase push down on the floor with an equal and opposite force $\vec{F}_{suitcase} = +m\vec{g}$ as shown in Figure (7).

This example of Newton's third law seems rather simplistic, because it is obvious that the force exerted by the suitcase is due to gravity pulling down on the suitcase. But in a few pages we will have a much more interesting example when we use the law to explain why the oceans have two tides per day.

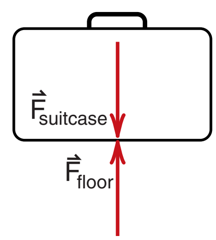


Figure 7
The suitcase and the floor exert equal and opposite forces on each other.

WEIGHT

If, instead of setting our suitcase on the floor, we set it on our bathroom scales, the suitcase will push down on the scales. ***We will define the force that the suitcase exerts on the scales*** as the ***weight*** of the suitcase. If the dial on the scales is correctly calibrated, the dial will give us a reading for the weight in newtons.

To calibrate the scales, we could go back to Chapter 6 and use the recoil definition of mass to make a number of replicas of the standard 1 kilogram mass. Then place one replica on the scales and mark where the pointer is over the dial. That 1 kg mass will exert a downward force $F_g = mg = 1\ kg \times 9.8\ m/sec^2 = 9.8\ newtons$. Label that mark 9.8 newtons as shown in Figure (8). Add a second mass, and the downward force becomes $F = 2\ kg \times 9.8\ m/sec^2 = 19.6\ newtons$. Label that point and keep adding 1 kg masses and labeling until you have covered the desired range of weights. In this way you will end up with properly calibrated scales.

One problem is that you will not find properly calibrated scales in the store if you are looking for metric scales. The scales will be marked in kilograms, showing a reading of "1 kilogram" when you place a 1 kg mass on the scales, etc. To get what physicists define as the weight of an object, you need to multiply the reading of the commercial scales by a factor of 9.8 to convert from kilograms to a force in newtons. Physicists may sound fairly picky, insisting that weight be defined as a force and measured in newtons rather than kilograms, but the next section helps explain why we do this.

Exercise 1

Explain how you would calibrate a set of bathroom scales if your laboratory were on the surface of the moon. To do this, use the results of Exercise (3-4) on page 3-17.

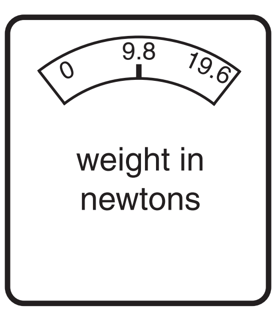


Figure 8
Bathroom scales calibrated using 1 kilogram masses.

Weightlessness and Free Fall

The newspapers often use the expression ***weightless*** to describe the condition of astronauts in orbit. Often people confuse weightlessness with the idea that the astronauts have escaped gravity. This is not true at all. We have seen that gravity had the same effect on John Glenn in orbit, as on our steel ball projectile. We do not think of the steel ball as having escaped gravity just because we dropped it.

Instead we say that until the ball hits the floor it is in ***free fall***. When a diver jumps off a diving board, the diver is in free fall until she hits the water. In the same sense the astronaut in orbit is in free fall. As we will see, the only difference is that as the astronaut falls, the earth's surface falls away just as fast, and the astronaut does not hit the earth.

The newspapers use of the expression 'weightless' is correct, if you use our definition of weight as the reading of the bathroom scales. Have an astronaut in orbit stand on the scales, and since both the astronaut and the scales are in free fall, there will be no force between them, the reading will be zero, and the astronaut will be weightless.

However the astronaut in orbit still has mass. The fact that the astronaut is in free fall has no effect on her mass. If we had calibrated our bathroom scales in kilograms, and then got a reading of zero kilograms in orbit, we would leave the wrong impression that the astronaut has no mass. On the other hand, a reading of zero newtons correctly implies that the astronaut is exerting no force on the scales.

NEWTON'S LAW OF GRAVITY

We have all heard the story about Newton discovering the law of gravity when hit on the head by an apple. The real significance of this story was his realization that the same laws of physics apply to heavenly bodies like the moon as well as to earthly objects like a falling apple. As we saw back in Chapter 5, both the falling apple and the moon were accelerating toward the center of the earth. In addition, the accelerations were related, the moon's acceleration being weaker than the apple's by the square of the distance to the center of the earth.

Explicitly we had

$$\boxed{\frac{a_{apple}}{a_{moon}} = \frac{r^2_{moon\ orbit}}{r^2_{earth}}} \quad \text{(Equation 5-15 on page 5-9)}$$

where r_{earth} and r_{moon} are the distances of the apple and moon from the center of the earth as seen in Figure (9).

Newton extended his observation about the moon, and proposed a law of gravitation that was assumed to apply to all objects in the universe. This law is known as Newton's ***Universal Law of Gravitation***, which can be stated as follows:

If we have two small masses m_1 and m_2 separated by a distance r as shown in Figure (10), then the forces between them are proportional to the product m_1m_2 of their masses, and inversely proportional to the square of the distance r between them. This can be written in an equation of the form

$$\boxed{\left|\vec{F}_g\right| = G\frac{m_1m_2}{r^2}} \qquad \textit{Newton's law of gravity} \qquad (10)$$

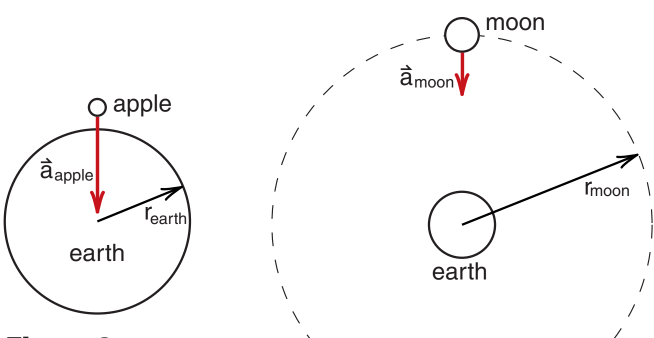


Figure 9
The same law of gravity causes both the apple and the moon to fall toward the earth.

The proportionally constant G is a number that must be determined by experiment.

Equation (10) is not the whole story, we must make several more points. First, and very important, is that the gravitational forces are always attractive, m_1 pulled directly toward m_2, and m_2 directly toward m_1. Second, the ***strengths of the forces are equal***. Even if m_2 is much larger than m_1, the force of m_2 on m_1 has the same strength as the force of m_1 on m_2. That is why we used the same symbol $\vec{F}_g$ for the two attractive forces in Figure (10).

With Newton's law of gravitation, we have an excellent, non trivial application of the third law of mechanics. Here we have an example of two objects interacting with each other, and the forces are equal in magnitude and oppositely directed ***as required by the third law***.

Newton's law of gravity is called the universal law of gravitation because Equation (10) is supposed to apply to all masses anywhere in the universe, with the same numerical constant G everywhere. G is called the ***universal gravitational constant***, and has the numerical value, in the MKS system of units

$$G = 6.67 \times 10^{-11}\frac{meter^3}{kg\ sec^2} \qquad \textit{universal gravitational constant} \qquad (11)$$

We will discuss shortly how this number was first measured.

Exercise 2

Combine Newton's second law $\vec{F} = m\vec{a}$ with the law of gravity $\left|\vec{F}_g\right| = Gm_1m_2/r^2$ and show that the dimensions for G in Equation (4) are correct.

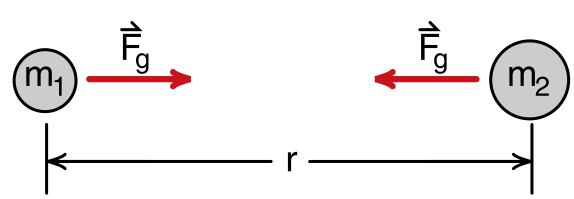


Figure 10
The gravitational force between small masses is proportional to the product of the masses, and inversely proportional to the square of the separation between them.

Big Objects

In our statement of Newton's law of gravity, we were careful to say that Equation (10) applied to two small objects. To be more explicit, we mean that the two objects m_1 and m_2 should be small in dimensions compared to the separation r between them. We can think of Equation (10) as applying to two ***point particles*** or ***point masses***.

What happens if one or both of the objects are large compared to their separation? Suppose, for example, that you would like to calculate the gravitational force between you and the earth as you stand on the surface of the earth. The correct way to do this is to realize that you are attracted, gravitationally, to every rock, tree, every single piece of matter in the entire earth as indicated in Figure (11). Each of these pieces of matter is pulling on you, and together they produce a net gravitational force $\vec{F}_g$ which is the force $m\vec{g}$ that we saw in our discussion of projectile motion.

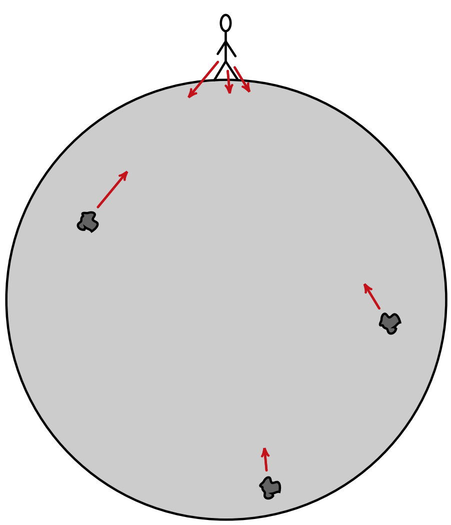


Figure 11
You are attracted to every piece of matter in the earth.

It appears difficult to add up all the individual forces exerted by every chunk of matter in the entire earth, to get the net force $\vec{F}_g$. Newton also thought that this was difficult, and according to some historical accounts, invented calculus to solve the problem. Even with calculus, it is a fairly complicated problem to add up all of these forces, but the result turns out to be very simple. ***For any uniformly spherical object, you get the correct answer in Newton's law of gravity if you think of all the mass as being concentrated at a point at the center of the sphere.*** (This result is an accidental consequence of the fact that gravity is a $1/r^2$ force, i.e., that it is inversely proportional to the square of the distance. We will have much more to say about this accident in later chapters.)

Since the earth is nearly a uniform spherical object, you can calculate the gravitational force between you and the earth by treating the earth as a point mass located at its center, 6,380 km or 4,000 miles below you, as indicated in Figure (12).

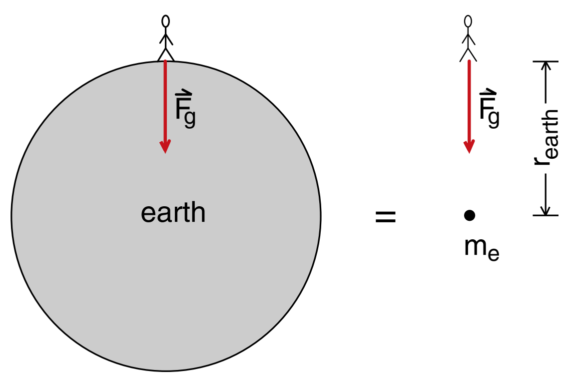


Figure 12
The gravitational force of the entire earth acting on you is the same as the force of a point particle with a mass equal to the earth mass, located at the earth's center, one earth radius below you.

Galileo's Observation

One of Galileo's great predictions was that, in the absence of air resistance, all projectiles should have the same acceleration no matter what their mass. Galileo's prediction leads to the striking result that, in a vacuum, a steel ball and a feather fall at the same rate. We can see that this is a consequence of Newton's second law combined with Newton's law of gravity.

Using the results of Figure (12), i.e., calculating $\vec{F}_g$ by replacing the earth by a point mass m_e located a distance r_e below us, we get

$$F_g = \frac{Gmm_e}{r_e^2} \qquad (12)$$

for the strength of the gravitational force on a particle of mass m at the surface of the earth. Combining this with Newton's second law

$$\vec{F}_g = m\vec{g} \quad \text{or} \quad F_g = mg \qquad (13)$$

we get

$$\cancel{m}g = \frac{G\cancel{m}m_e}{r_e^2} \qquad (14)$$

The important result is that the particle's mass m cancels out of Equation (14), and we are left with the formula

$$\boxed{g = \frac{Gm_e}{r_e^2}} \qquad (15)$$

for the acceleration due to gravity. We note that g depends on the earth mass m_e , the earth radius r_e, and the universal constant G, but ***not on the particle's mass m***. Thus objects of different mass should have the same acceleration.

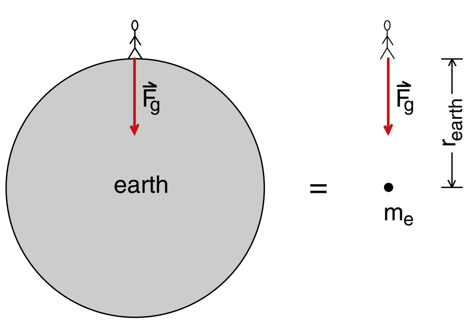


Figure 12 repeated

Galileo Was Right!

By WILLIAM HINES

HOUSTON — Galileo was right and Apollo 15 Astronaut David R. Scott can prove it.

The 17-th century Italian mathematician believed contrary to then prevailing scientific opinion that gravity exerts an equal influence on all things, regardless of their size, shape or weight.

He even went so far as to say that if it weren't for the resistance offered by air, a cannonball and a feather would fall at the same speed.

It wasn't possible for Galileo to demonstrate the truth of that assertion, although legend says he performed a compromise experiment by dropping a large iron ball and a small one off the leaning tower of Pisa and that they hit the ground at the same instant.

Just before getting back into the Lunar Landing Craft Falcon fcı takeoff from the Moon, Scott demonstrated that Galileo was right.

Holding a metal geological hammer in his right hand and a feather ("A falcon feather," he explained) in his right, Scott faced the Apollo television camera and released the two objects.

Falling slowly in the eak gravity of the airless Moon, the hammer and the feather reached the surface at precisely the same instant.

"How about that?" the delighted Scott remarked. "Mr. Galileo was correct in his findings."

Exercise 3

As shown in Figure (9) repeated here, gravity is causing the falling apple and the moon to accelerate toward the center of the earth. We have just calculated that the apple's acceleration is $a_{apple} = g = Gm_{earth}/r^2_{earth}$.

a) Calculate the magnitude of the moon's acceleration a_{moon} toward the center of the earth.

b) Calculate the ratio a_{apple}/a_{moon} . How does the result compare with Equation (5-15)?

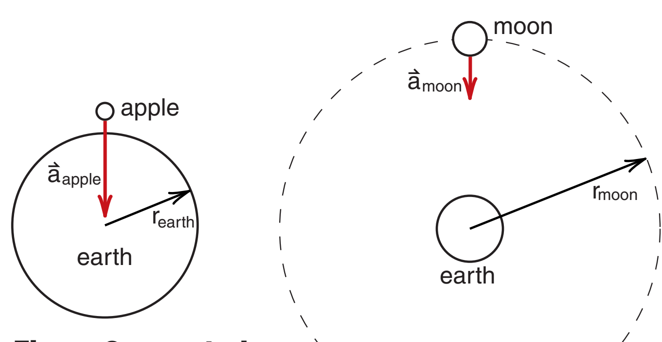


Figure 9 repeated
The same law of gravity causes both the apple and the moon to fall toward the earth.

THE CAVENDISH EXPERIMENT

A key feature of Newton's law of gravitation is that all objects attract each other via gravity. Yet in practice, the only gravitational force we ever notice is the force of attraction to the earth. What about the gravitational force between two students sitting beside each other, or the force between your two fists when you hold them close to each other? The reason that you do not notice these forces is that the gravitational force is incredibly weak, weak compared to other forces that hold you, trees, and rocks together. Gravity is so weak that you would never notice it, except for the fact that you are on top of a huge hunk of matter called the earth. The earth's mass is so great that, even with the weakness of gravity, the resulting force between you and the earth is big enough to hold you down to the surface.

The gravitational force between two reasonably sized objects is not so small that it cannot be detected, it just requires a very careful experiment that was first performed by Henry Cavendish in 1798. In the Cavendish experiment, two small lead balls are mounted on the end of a light rod. This rod is then suspended on a fine glass fiber as shown in Figure (13a).

As seen in the top view in Figure (13b), two large lead balls are placed near the small ones in such a way that the gravitational force between each pair of large and small balls will cause the rod to rotate in one direction. Once the rod has settled down, the large lead balls are moved to the position shown in Figure (13c). Now the gravitational force causes the rod to rotate the other way. By measuring the angle that the rod rotates, and by measuring what force is required to rotate the rod by this angle, one can experimentally determine the strength of the gravitational force $\vec{F}_g$ between the balls. Then by using Newton's law of gravity

$$F_g = G\frac{m_1 m_2}{r^2}$$

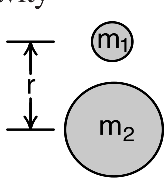
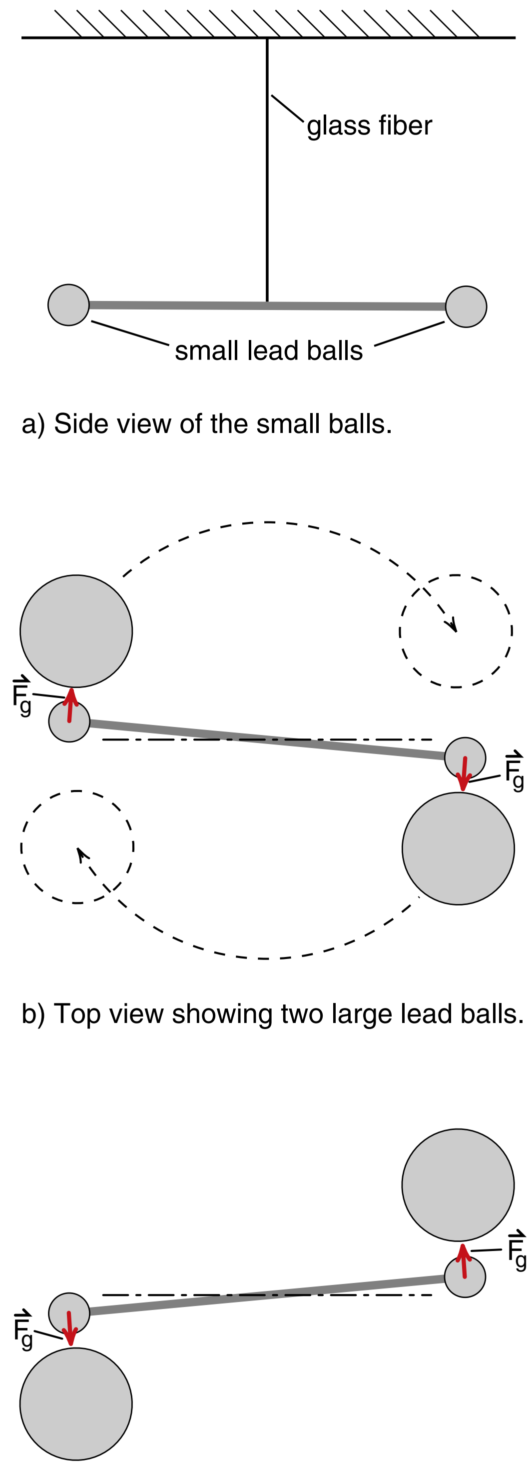


Figure 13d

applied to Figure (13d), one can solve for G in terms of the known quantities F_g, m_1, m_2 and r^2. This was the way that Newton's universal constant G, given in Equation (10) was first measured.



a) Side view of the small balls.

b) Top view showing two large lead balls.

c) Top view with large balls rotated to new position.

Figure 13
The Cavendish experiment. By moving the large lead balls, the small lead balls are first pulled one way, then the other. By measuring the angle of rotation of the stick holding the small balls, one can determine the gravitational force $\vec{F}_g$.

"Weighing" the Earth

Once you know G, you can go back to Equation (15) for the acceleration g due to gravity, and solve for the earth mass m_e to get

$$m_e = \frac{g r_e^2}{G} = \frac{9.8\ \text{m/sec}^2 \times (6.37 \times 10^6 \text{m})^2}{6.67 \times 10^{-11} \text{m}^3/\text{kg sec}^2}$$

$$= 6.0 \times 10^{24}\ \text{kg} \qquad (16)$$

As a result, Cavendish was able to use his value for G to determine the mass of the earth. This was the first accurate determination of the earth's mass, and as a result the Cavendish experiment became known as the experiment that "weighed the earth".

Exercise 4

The density of water is 1 gram/cm^3. The average density of the earth's outer crust is about 3 times as great. Use Cavendish's result for the mass of the earth to decide if the entire earth is like the crust. (Hint —the volume of a sphere of radius r is $4/3\pi r^3$). Relate your result to what you have read about the interior of the earth.

Inertial and Gravitational Mass

The fact that, in the absence of air resistance, all projectiles have the same acceleration—the fact that the m's canceled in Equation (14), has a deeper consequence than mere coincidence. In Newton's second law, the m in the formula $\vec{F} = m\vec{a}$ is the mass defined by the recoil definition of mass discussed in Chapter 6. Called ***inertial mass***, it is the concept of mass that we get from the law of conservation of linear momentum.

In Newton's law of gravity, the projectile's mass m in the formula $F_g = Gmm_e/r_e^2$ is what we should call the ***gravitational mass*** for it is defined by the gravitational interaction. It is the experimental observation that the m's cancel in Equation (14), the observation that all projectiles have the same acceleration due to gravity, that tells us that the inertial mass is the same as gravitational mass. This equivalence of inertial and gravitational mass has been tested with extreme precision, to one part in a billion by Etvös in 1922, and to even greater accuracy by R. H. Dicke in the 1960s.

EARTH TIDES

An aspect of Newton's law of gravity that we have not said much about is the fact that gravity is a mutual attraction. As we mentioned, two objects of mass m_1 and m_2 separated by a distance r, attract ***each other*** with a gravitational force of magnitude $F_g = Gm_1m_2/r^2$. The point we want to emphasize now is that the force on ***each*** particle has the same strength F_g.

Let us apply this idea to you, here on the surface of the earth. Explicitly, let us assume that you have just jumped off a high diving board as illustrated in Figure (14), and have not yet hit the water. While you are falling, the earth's gravity exerts a downward force $\vec{F}_g$ which produces your downward acceleration $\vec{g}$.

According to Newton's law of gravity, you are exerting an equal and opposite gravitational force $\vec{F}_g$ on the earth. Why does nobody talk about this upward force you are exerting on the earth? The answer, shown in the following exercise, is that even though you are pulling up on the earth just as hard as the earth is pulling down on you, the earth is so much more massive that your pull has no detectable effect.

Exercise 5

Assume that the person in Figure (14) has a mass of 60 kilograms. The gravitational force he exerts on the earth causes an upward acceleration of the earth a_{earth}. Show that $a_{earth} = 10^{-22}\text{m/sec}^2$.

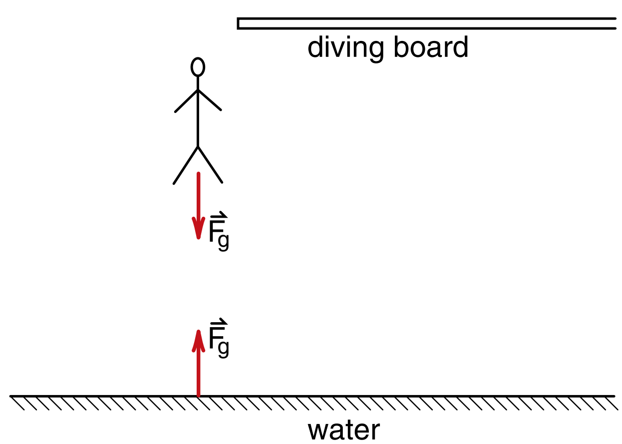


Figure 14
As you fall toward the water, the earth is pulling down on you, *and* ***you are pulling up on the earth. The two forces are of equal strength.***

More significant than the force of the diver on the earth is the force of the moon on the earth. It is well known that the ocean tides are caused by the moon’s gravity acting on the earth. On the night of a full moon, high tide is around midnight when the moon is directly overhead. The time of high tide changes by about an hour a day in order to stay under the moon.

The high tide under the moon is easily explained by the idea that the moon’s gravity sucks the ocean water up into a bulge under the moon. As the earth rotates and we pass under the bulge, we see a high tide. This explains the high tide at midnight on a full moon.

The problem is that there are 2 high tides a day about 12 hours apart. The only way to understand two high tides is to realize that there are two bulges of ocean water, one under the moon and one on the opposite side of the earth, as shown in Figure (15). In one 24 hour period we pass under both bulges.

Why is there a bulge on the back side? Why isn’t the water all sucked up into one big bulge underneath the moon?

The answer is that the moon’s gravity not only pulls on the earth’s water, but on the earth itself. The force of gravity that the moon exerts on the earth is just the same strength as the force the earth exerts on the moon. Since the earth is more massive, the effect on the earth is not as great, but it is noticeable. The reason for the second bulge of water on the far side of the earth is that the ***center of the earth is closer to the moon than the water on the back side, and therefore accelerates more rapidly toward the moon than the water on the back side***. The water on the back side gets left behind to form a bulge.

The fact that there are two high tides a day, the fact that there is a second bulge on the back side, is direct experimental evidence that the earth is accelerating toward the moon. It is direct evidence that the moon’s gravity is pulling on the earth, just as the earth’s gravity is pulling on the moon.

As a consequence of the earth’s acceleration, the moon is not traveling in a circular orbit centered precisely on the center of the earth. Instead both the earth and the moon are traveling in circles about an axis point located on a line joining the earth’s and moon’s centers. This axis point is located inside the earth about 1/4 of the way down below the earth's surface as shown in Figure (16).

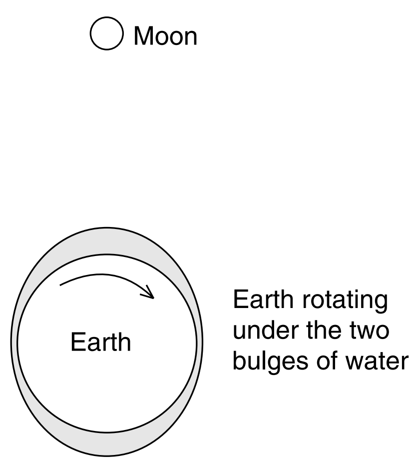


Figure 15
The two ocean bulges cause two high tides per day.

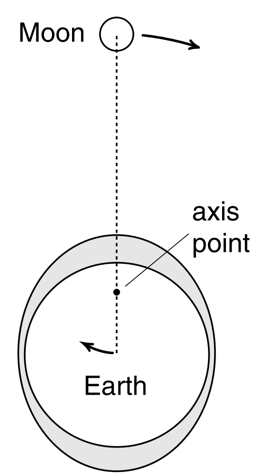


Figure 16
Both the earth and the moon travel in circular orbits about an axis point located about 1/4 of the way down below the earth's surface.

SATELLITE MOTION

In various parts of the text, we have used satellite motion to develop concepts like acceleration that are key to understanding Newtonian mechanics. In Chapter 3, after using strobe photographs to determine the acceleration of a steel ball projectile, we used the same graphical techniques to show that John Glenn, in low orbit above the lab, had essentially the same acceleration vector. This led to the picture that both the steel ball and John Glenn were falling toward the center of the earth. The reason that Glenn did not hit the earth is that the earth was falling away at the same rate as shown in Figure (5-10) reproduced below. In Chapter 3, we also showed Newton's famous sketch, Figure (3-22), that compares projectile and satellite motion.

Here we would like to change the approach. Suppose that instead of beginning with the analysis of strobe photographs, we began the study of physics by being given Newton's second law and the law of gravitation. We are then asked to use these laws to derive as much as we can about satellite motion.

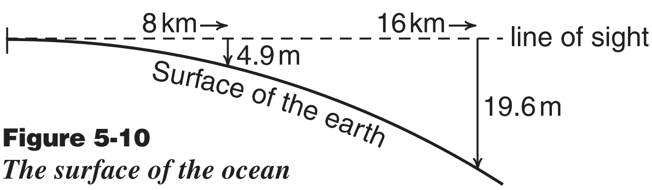


Figure 5-10
The surface of the ocean falls away as you look out to sea.

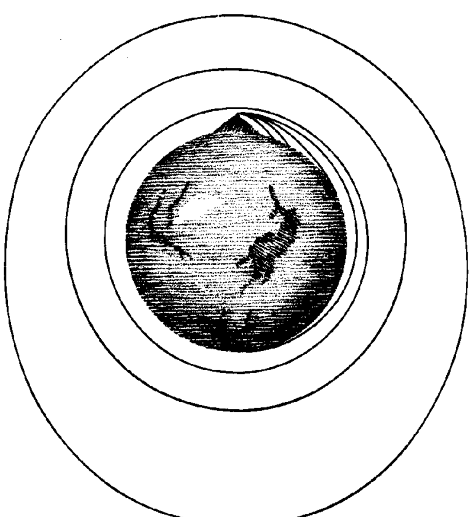


Figure 3-22
Newton's sketch, showing that the difference between projectile and satellite motion is that satellites travel farther. Both are accelerating toward the center of the earth.

As a sample exercise, we will calculate the period of an earth satellite in an orbit of radius r. (This is more or less a review of the steps you should have followed for Exercise 3a.)

The first step in solving a Newton's law problem is to draw a careful sketch of the situation, showing the forces involved. Our sketch is shown in Figure (17). The simple feature of satellite motion is that only one force is involved. It is the gravitational force $\vec{F}_g$ that the earth exerts on the satellite. From Newton's law of gravitation, $\vec{F}_g$ is directed toward the center of the earth and has a magnitude

$$F_g = G\frac{m_{earth}m_{satellite}}{r^2} \qquad (17)$$

Since $\vec{F}_g$ is the total force acting on the satellite, we can use Newton's second law, $\vec{F}_{total} = m\vec{a}$ to get

$$\vec{F}_g = m_{earth}\vec{a}_{satellite} \qquad (18)$$

Because $\vec{F}_g$ points toward the center of the earth, the satellite's acceleration $\vec{a}_{satellite}$ must also point there.

Since both $\vec{F}_g$ and $\vec{a}_{satellite}$ point in the same direction, we can write Equation (18) in terms of the magnitudes of these two vectors

$$F_g = m_{earth}a_{satellite} \qquad (18a)$$

We can now combine Equations (17) and (18a) to get

$$F_g = G\frac{m_{earth}\cancel{m_{satellite}}}{r^2} = \cancel{m_{satellite}}a_{satellite} \qquad (19)$$

A very important thing happens in Equation (19). The satellite mass $m_{satellite}$ cancels and we are left with

$$\boxed{a_{satellite} = G\frac{m_{earth}}{r^2}} \qquad (20)$$

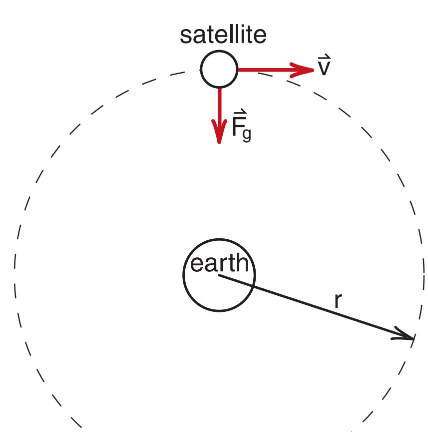


Figure 17
Sketch for calculating the satellite acceleration.

Period of Satellite

Now that we have a formula for the satellite's acceleration vector $\vec{a}_{satellite}$, we can study circular orbits using the formula

$$a = \frac{v^2}{r} \qquad \text{(5-8, page 5-4)}$$

for the magnitude of the acceleration of a particle moving at a constant speed in a circle of radius r. As we saw, $\vec{a}$ is directed toward the center of the circle, which for an earth satellite is the center of the earth.

For circular orbits, we can combine Equation (20) and (5-8) to solve for the speed $v_{satellite}$. We get

$$a_{satellite} = G\frac{m_{earth}}{r^2} = \frac{v^2_{satellite}}{r} \qquad (21)$$

One of the r's cancels and we are left with

$$v_{satellite} = \sqrt{G\frac{m_{earth}}{r}} \qquad (22)$$

To calculate the period T of the satellite, we note that T is the time the satellite takes to travel around the circumference $2\pi r$ of its orbit, at a speed $v_{satellite}$. Keeping careful track of dimensions, we have

$$T\frac{sec}{orbit} = 2\pi r\frac{meters}{orbit} \times \frac{1}{v\frac{meters}{sec}} = \frac{2\pi r}{v}\frac{sec}{orbit} \qquad (23)$$

The dimensions meters cancel and we are left with seconds/orbit for T.

Using Equation (22) for $v_{satellite}$ in (23) gives

$$T \text{ seconds} = 2\pi r\sqrt{\frac{r}{Gm_{earth}}} \text{ seconds} \qquad (24)$$

Since the result is a bit messy, let us see if it gives the right value for Glenn's orbit. Glenn's orbital radius r is the earth radius plus the height of his orbit, namely $(6.38 \times 10^6 + 260{,}000)$ meters = 6.64×10^6 meters.

Using this value of r in Equation (24) gives

$$T_{sec} = 2\pi \times 6.64 \times 10^6 m \sqrt{\frac{6.64 \times 10^6 m}{6.67 \times 10^{11}\frac{m^3}{kg\ sec^2} \times 5.98 \times 10^{24} kg}}$$

$$= 4.17 \times 10^7 \text{meters} \sqrt{1.66 \times 10^{-8}\frac{\text{seconds}^2}{\text{meters}^2}}$$

We get

$$T_{seconds} = 5380 \text{ seconds}$$

Converting to minutes gives

$$T_{Glenn} = \frac{5380 \text{ seconds}}{60 \text{ sec/min}} = 89.6 \text{ minutes} \qquad (25)$$

Which is the correct answer for the period of Glenn's orbit.

Exercise 6

Geosynchronous satellites are satellites that orbit over the equator with a period T equal to 24 hours. As a result the satellites stay over the same point in the earth and appear to be stationary in the sky. Such satellites are used for various kinds of communication and for broadcasting satellite TV. Use the results of our previous calculations to determine how high above the surface of the earth these satellites are located.

KEPLER'S LAWS OF PLANETARY MOTION

When Newton developed his law of gravity, he had much more to go on than the comparison of the acceleration of the moon and the apple toward the earth. There had been centuries of study of the motion of the planets and stars, culminating in Kepler's three laws of planetary motion.

As we will show, Kepler's first two laws clearly imply that the planet accelerates toward the sun, and that this acceleration drops off as the inverse square of the distance to the sun. We will show this by creating a strobe photograph of the motion of a hypothetical planet and then do a graphical analysis of the orbit. This is an exercise we could have done back in Chapter 3.

But before we go into the details of Kepler's laws, we would like to take a brief look at the historical background that led to Kepler's discovery.

BACKGROUND FOR KEPLER'S LAWS

To get a feeling for the problems involved in studying planetary motion, imagine that you were given the job of going outside, looking at the sky, and figuring out how celestial objects moved. The easiest to start with is the moon, which becomes full again every four weeks. On closer observation you would notice that the moon moved past the background of the apparently fixed stars, returning to its original position in the sky every 27.3 days. Since the diameter of the moon does not change much, you might then conclude that the moon is in a circular orbit about the earth, with a period of 27.3 days.

The time it takes the moon to return to the same point in the sky is not precisely equal to the time between full moons. A full moon occurs when the sun, earth, and moon are in alignment. If the sun itself appears to move relative to the fixed stars, the full moons will not occur at precisely the same point, and the time between full moons will not be exactly the time it takes the moon to go around once.

To study the motion of the sun past the background of the fixed stars is more difficult because the stars are not visible when the sun is up. One way to locate the position of the sun is to observe what stars are overhead at "true" midnight, half way between dusk and dawn. The sun should then be located on the opposite side of the sky. (You also have to correct for the north/south position of the sun.) After a fair amount of observation and calculations, you would find that the sun itself moves past the background of the fixed stars, returning to its starting point once a year.

From the fact that the sun takes one year to go around the sky, and the fact that its apparent diameter remains essentially constant, you might well conclude that the sun, like the moon, is traveling in a circular orbit about the earth. This was the accepted conclusion by most astronomers up to the time of Nicolaus Copernicus in the early 1500s AD.

If you start looking at the motion of the planets like Mercury, Venus, Mars, Jupiter, and Saturn, all easily visible without a telescope, the situation is more complicated. Mars, for example, moves in one direction against the background of the fixed stars, then reverses and goes backward for a while, then forward again as shown in Figure (18). None of the planets has the simple uniform motion seen in the case of the moon and the sun.

After a lot of observation and the construction of many plots, you might make a rather significant discovery. You might find what the early Greek astronomers learned, namely that if you assume that the planets Mercury, Venus, Mars, Jupiter, and Saturn travel in circular orbits about the sun, while the sun is traveling in a circular orbit about the earth, then you can explain all the peculiar motion of the planets. This is a remarkable simplification and compelling evidence that there is a simple order underlying the motion of celestial objects.

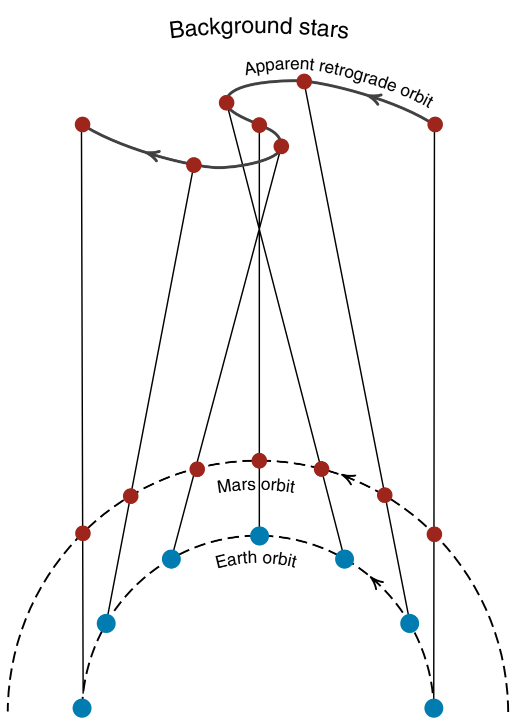


Figure 18
Retrograde motion of the planet Mars. Modern view of why Mars appears to reverse its direction of motion for a while.

One of the features of astronomical observations is that they become more accurate as time passes. If you observe the moon for 100 orbits, you can determine the average period of the moon nearly 100 times more accurately than from the observation of a single period. You can also detect any gradual shift of the orbit 100 times more accurately.

Even by the time of the famous Greek astronomer Ptolemy in the second century AD, observations of the positions of the planets had been made for a sufficiently long enough time that it had become clear that the planets did not travel in precisely circular orbits about the sun. Some way was needed to explain the non circularity of the orbits.

The simplicity of a circular orbit was such a compelling idea that it was not abandoned. Recall that the apparently peculiar motion of Mars could be explained by assuming that Mars traveled in a circular orbit about the sun which in turn traveled in a circular orbit about the earth. By having circular orbits centered on points that are themselves in circular orbits, you can construct complex orbits. By choosing enough circles with the correct radii and periods, you can construct any kind of orbit you wish.

Ptolemy explained the slight variations in the planetary orbits by assuming that the planets traveled in circles around points which traveled in circles about the sun, which in turn traveled in a circle about the earth. The extra cycle in this scheme was called an ***epicycle***. With just a few epicycles, Ptolemy was able to accurately explain all observations of planetary motion made by the second century AD.

With 1500 more years of planetary observations, Ptolemy's scheme was no longer working well. With far more accurate observations over this long span of time, it was necessary to introduce many more epicycles into Ptolemy's scheme in order to explain the positions of the planets.

Even before problems with Ptolemy's scheme became apparent, there were those who argued that the scheme would be simpler if the sun were at the center of the solar system and all the planets, including the earth, moved in circles about the sun. This view was not taken seriously in ancient times, because such a scheme would predict that the earth was moving at a tremendous speed, a motion that surely would be felt. (The principle of relativity was not understood at that time.)

For similar reasons, one did not use the rotation of the earth to explain the daily motion of sun, moon, and stars. That would imply that the surface of the earth at the equator would be moving at a speed of around a thousand miles per hour, an unimaginable speed!

In 1543, Nicolaus Copernicus put forth a detailed plan for the motion of the planets from the point of view that the sun was the center of the solar system and that all the planets moved in circular orbits about the sun. Such a theory not only conflicted with common sense about feeling the motion of the earth, but also displaced the earth and mankind from the center of the universe, two results quite unacceptable to many scholars and theologians.

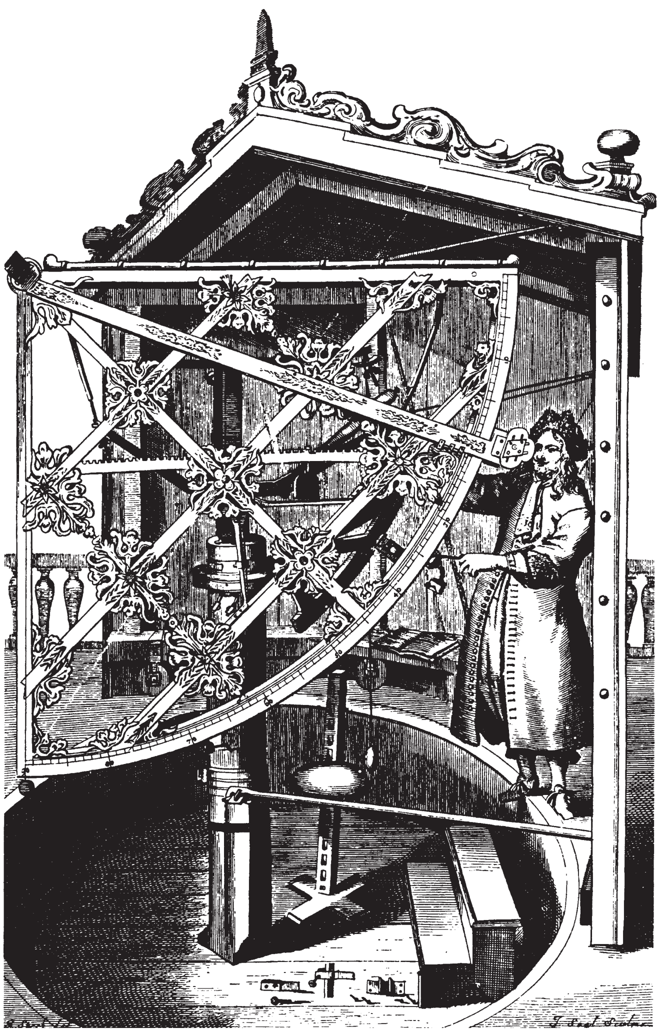


Figure 19
Tycho Brahe's apparatus.

Copernicus' theory was not quite as simple as it first sounds. Because of the accuracy with which planetary motion was know by 1543, it was necessary to include epicycles in the planetary orbits in Copernicus' model.

Starting around 1576, the Dutch astronomer Tycho Brahe made a series of observations of the planetary positions that were a significant improvement over previous measurements. This work was done before the invention of the telescope, using apparatus like that shown in Figure (19). Tycho Brahe did not happen to believe in the Copernican sun-centered theory, but that had little effect on the reason for making the more accurate observations. Both the Ptolemaic and Copernican systems relied on epicycles, and more accurate data was needed to improve the predictive power of these theories.

Johannes Kepler, a student of Tycho Brahe, started from the simplicity inherent in the Copernican system, but went one step farther than Copernicus. Abandoning the idea that planetary motion had to be described in terms of circular orbits and epicycles, Kepler used Tycho Brahe's accurate data to look for a better way to describe the way planets move. The result was his three laws of planetary motion.

KEPLER'S FIRST LAW

Kepler's first law is that ***the planets move in elliptical orbits with the sun at one focus***. In Figure (20) we show you how to draw an ellipse. Pound two nails into a board and tie a string loosely between the nails. Then stretch the string taut with a pencil as shown in Figure (20), and move the pencil all the way around the nails, keeping the string taut. The resulting curve is an ellipse, and the nail holes are the focuses of the ellipse. If this curve represented the path of a planet around the sun, then the center of the sun would be at one of the nail holes.

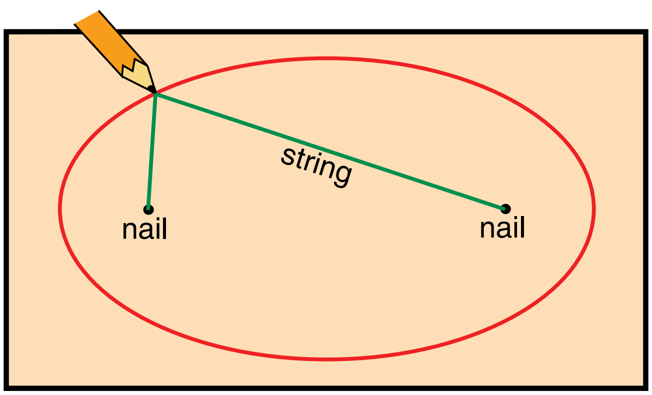


Figure 20
Ellipse constructed with two nails and a string. In colonial times, this technique was used to make the arches over carriage house doors.

KEPLER'S SECOND LAW

Kepler's second law states that ***the radius vector from the sun to the planet sweeps out equal areas in equal times***. A radius vector $\vec{R}$ is a vector whose tail is fixed at the origin and whose tip follows the motion of the object. In Figure (21) we placed the sun at one focus, called that the origin, and had the tip of the radius vector move around in an elliptical orbit. The large black dots are located at points along the orbit so that the same area is swept out as the tip of $\vec{R}$ moves from dot to dot. We show three of these areas, starting at the points labeled (a), (b), and (c). In Exercise (7) we will have you check that these shaded areas are equal.

The first thing we notice is that the planet travels a much greater distance from points (a) to (a1) than from points (c) to (c1). Since the planet has to cover these two distances in the same length of time, the planet has to travel much faster when it is in close to the sun than when it is at the distant part of the orbit. If the planet were in a circular orbit, its speed would be the same all the way around. In an elliptical orbit, there can be a considerable change in speed.

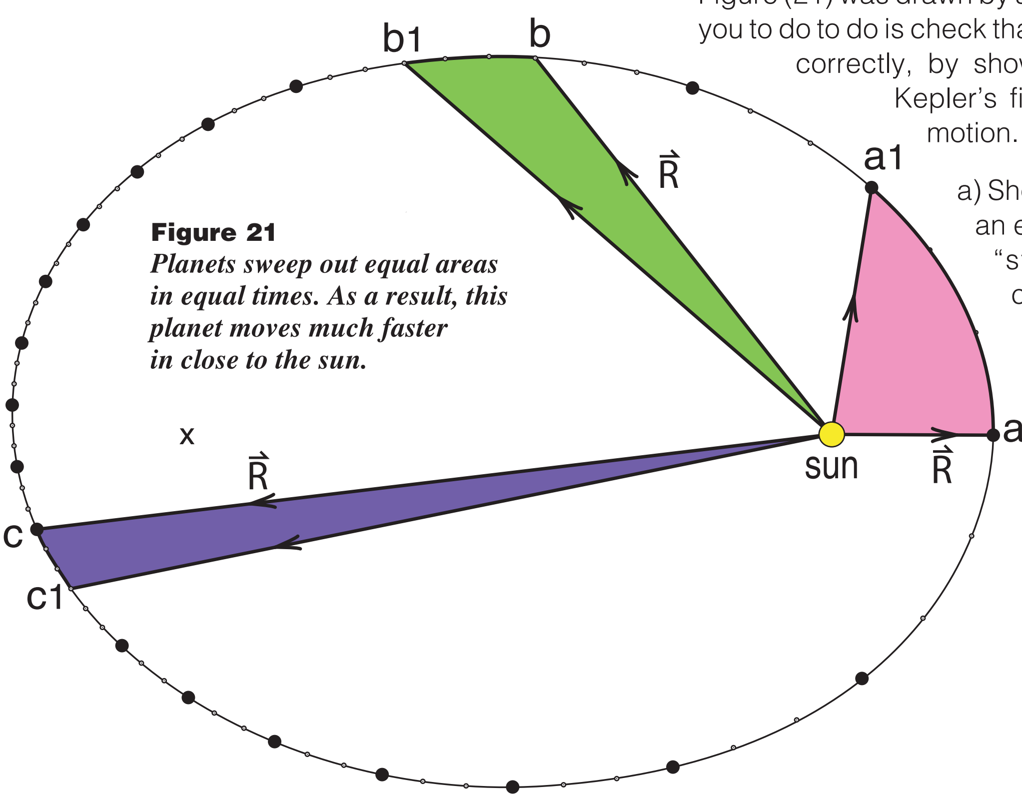


Figure 21
Planets sweep out equal areas in equal times. As a result, this planet moves much faster in close to the sun.

Exercise 7

Figure (21) was drawn by a computer. What we want you to do to do is check that the computer did the job correctly, by showing that the plot obeys Kepler's first two laws of planetary motion.

a) Show that the orbit is actually an ellipse by showing that the "string length" does not change.

b) Show that the three shaded areas in Figure (25) are equal to each other.

STROBE PHOTOGRAPH OF PLANETARY MOTION

Because the big dots in Figure (21) are located at equal time intervals along the planet's orbit, we can treat this figure as a strobe photograph of the planet's motion. This is similar to what we did for John Glenn's orbit.

When we analyzed Glenn's orbit we discovered that Glenn was accelerating toward the center of the earth with nearly the same acceleration vector as our steel ball projectile on the surface of the earth. This told us that gravity was having about the same effect on Glenn and the steel ball.

Now that we have a strobe photograph of planetary motion, we can study the effect the sun's gravity has on the motion of a planet. Explicitly, we can use our graphical analysis to determine the planet's acceleration as it moves around the sun.

Figure (22) is the same basic diagram as Figure (21), except we are using the small dots to show the location of the planet at a shorter time interval. (The big dots were located at every third small dot.)

In order to make Figure (22) appear more like a planetary orbit, we will assume that the time interval between small dots (our strobe time) is one month. Since there are 66 small dots, such a planet would be taking 66 months or 5.5 years to orbit the sun.

For a distance scale, we will assume that when our planet is at its closest approach to the sun, at its ***perigee*** at point (a), it is the same distance from the sun as the earth is. That distance is called an ***Astronomical Unit*** or ***AU,*** and is 93 million miles, 150 km, or 8 light minutes. Thus the radius vector at point (a) has a length $R_a = 1$ AU.

We chose to analyze the planet's acceleration at four specific points along the orbit. The first is at point (a) where the planet's radius vector has a length $R_a = 1$ AU. We chose point (b) out where the radius vector has a length $R_b = 3$AU. Point (c) is where $R_c = 5$ AU. And at point (d), the radius vector is back to $R_d = 3$AU again.

When we analyzed Glenn's orbit back in Figure (3-21), we used the same value of Δt to analyze the acceleration at three different positions. If we did the same thing, keep using the same value of Δt to analyze the planet's acceleration out at points (b), (c) and (d), our displacement vectors $\vec{s}_1$ and $\vec{s}_2$ would get so short that the graphical work would not be accurate. What we have done instead is to use longer values of Δt out at the more distant points. At points (b) and (d) we chose $\Delta t = 3$, which means $\vec{s}_1$ and $\vec{s}_2$ are three month displacements. Out at point (c), we chose five month displacement vectors, using $\Delta t = 5$.

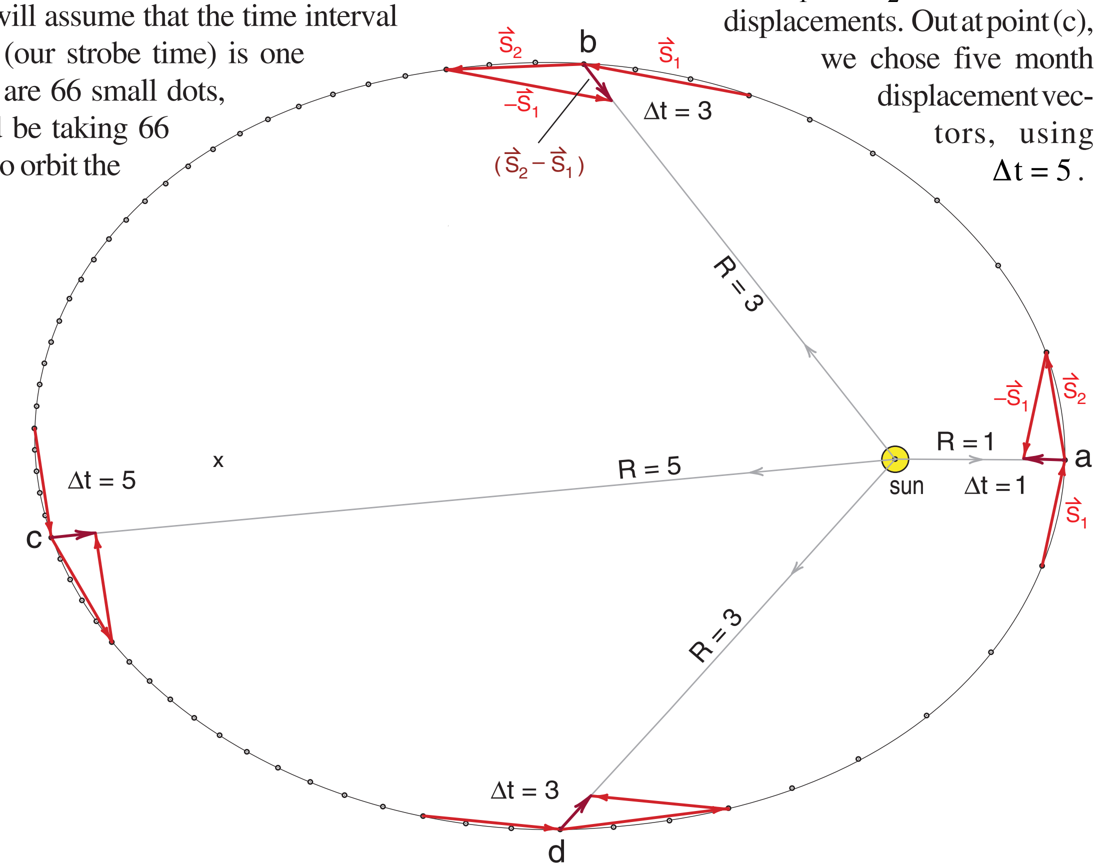
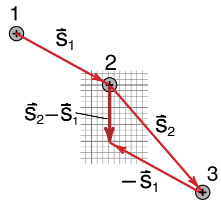


Figure 22
Simulated strobe photograph of planetary motion constructed using Kepler's laws.

As in all of our strobe photographs, the acceleration vector $\vec{A}$ is given by Equation (3-11), Page 3-14, as



$$\vec{A} = \frac{\vec{s}_2 - \vec{s}_1}{\Delta t^2}$$

where at each point we are calling $\vec{s}_1$ the entering displacement vector and $\vec{s}_2$ the exiting one.

In Figure (22) we have drawn the vectors $\vec{s}_1$ and $\vec{s}_2$ at the four points (a,b, c, d), and then in each case constructed the difference vector $(\vec{s}_2 - \vec{s}_1)$. Perhaps the most striking feature of Figure (22) is that ***all the difference vectors point straight at the sun***. Thus the planet is always accelerating toward the sun.

We had a similar effect for Glenn's orbit, but now we see that the gravitational acceleration is toward the sun even for an elliptical orbit where the planet's speed has a considerable variation. (If Glenn were fired into an elliptical orbit, his acceleration vector would still point to the center of the earth just as it does in a circular orbit.)

Magnitude of the Acceleration

The magnitude (A) of the planet's acceleration at the various points is given by the formula

$$A = \frac{\left|\vec{s}_2 - \vec{s}_1\right|}{\Delta t^2} \qquad (3\text{-}11)$$

where $\left|\vec{s}_2 - \vec{s}_1\right|$ is the length of the difference vector. A careful inspection of Figure (22) shows that all four difference vectors have the same length which we will call (s)

$$s \equiv \left|\vec{s}_2 - \vec{s}_1\right| \qquad \textit{for all four positions a, c, e \& g} \qquad (26)$$

As a result, our formula for the magnitude of the planet's acceleration becomes

$$A = \frac{s}{\Delta t^2} \qquad (27)$$

where it is the value of Δt that we changed as the planet went away from the sun. The explicit values of A at points (a), (b), and (c) are

$$A_a = \frac{s}{\Delta {t_a}^2} = \frac{s}{1^2} = s$$

$$A_b = \frac{s}{\Delta {t_b}^2} = \frac{s}{3^2} = \frac{s}{9}$$

$$A_c = \frac{s}{\Delta {t_c}^2} = \frac{s}{5^2} = \frac{s}{25} \qquad (28)$$

Thus we see, for example, that the planet's acceleration is 25 times weaker out at point (c) than it was at perigee, point (a).

The Inverse Square Law

In constructing Figure (22), we made what may have seemed like a rather arbitrary choice. At point (a) where R = 1, we chose $\Delta t = 1$. At points (b) and (d) where R = 3, we chose $\Delta t = 3$. Out at point (c) where R = 5, we chose $\Delta t = 5$. In other words we arbitrarily chose Δt equal to the distance R in the units we are using. If we replace Δt by R in our formula for the magnitude of the planet's acceleration at these four points, we get

$$\boxed{A = \frac{s}{\Delta t^2} = \frac{s}{R^2}} \qquad \textit{for all four positions a, b, c \& d} \qquad (29)$$

Thus we see, by an analysis of the planet's acceleration at four different points, that the planet's acceleration due to gravity drops off as the inverse square of the distance from the planet to the sun. This $1/R^2$ behavior of the gravitational acceleration is a direct consequence of Kepler's first two laws of motion. Issac Newton knew this when he formulated his law of gravity.

Footnote

When our hypothetical planet is at perigee in Figure (22), it is at the same distance from the sun as the earth (1 AU), but it is traveling 1.3 times faster than the earth. That is why it moves in an extended ellipse that goes out past Mars, into the asteroid belt. It goes half way to Jupiter's 10 AU radius circular orbit.

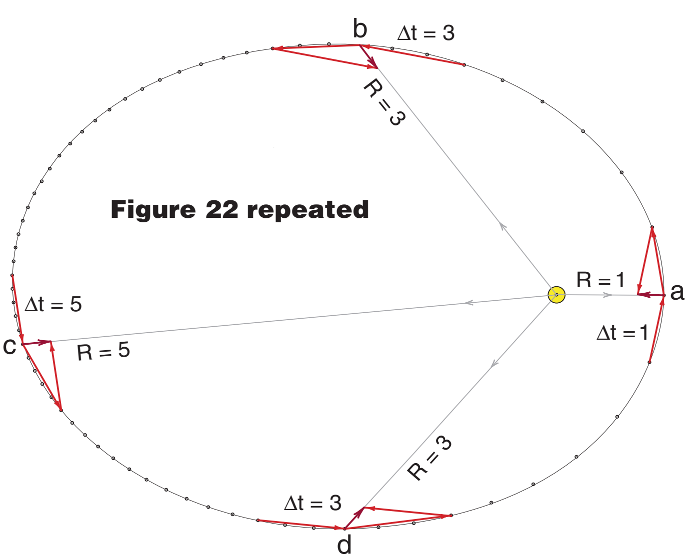


Figure 22 repeated

KEPLER'S THIRD LAW

Kepler's third law relates the period of the planet in its elliptical orbit to the width of the ellipse. In describing ellipses, one usually refers to the semi major axis R shown in Figure (23). The explicit statement of the third law was that the ratio of the cube of the semi major axis R to the square of the period T, that is, the ratio R^3/T^2, was the same for all planets. What we want to do now is to use Newton's laws to figure out what Kepler's third law is trying to tell us. We will calculate the ratio R^3/T^2 for circular orbits.

Earlier we used Newton's laws to calculate the period T of an earth satellite in a circular orbit of radius r. The result was

$$T \text{ sec} = 2\pi r\sqrt{\frac{r}{Gm_{earth}}} \text{ sec} \qquad \text{(24 repeated)}$$

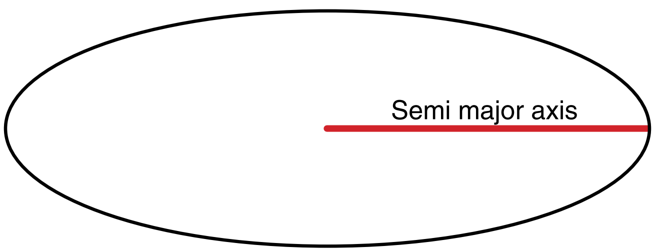


Figure 23
Definition of the semi major axis of an ellipse.

The same formula applies to a planet in orbit about the sun if we replace m_{earth} by m_{sun}. Doing that and squaring the equation gives

$$T^2 = 4\pi^2 r^2 \left(\frac{r}{Gm_{sun}}\right) = \frac{4\pi^2 r^3}{Gm_{sun}} \qquad (30)$$

Dividing through by r^3 gives

$$\frac{T^2}{r^3} = \frac{4\pi^2}{Gm_{sun}}$$

or turning the equation upside down

$$\boxed{\frac{r^3}{T^2} = m_{sun}\left[\frac{G}{4\pi^2}\right]} \qquad \textit{for a planet in a circular orbit about the sun} \qquad (31)$$

Since G is a universal constant, you can see that the ratio r^3/T^2 depends only on the mass of the sun, and therefore should be the same for all the planets.

It is interesting to note that once Cavendish had measured the value of G, it became possible to use the observed value of r^3/T^2 to measure the mass m_{sun} of the sun. In colloquial, and somewhat inaccurate terms, Kepler's third law allows us to weigh the sun.

MODIFIED GRAVITY AND GENERAL RELATIVITY

In the Essay 2 following this chapter, we describe how Newton's step-by-step method of predicting motion makes it rather easy to go from one kind of physics problem to another. In particular, we outline the steps that allow us to change a procedure for predicting projectile motion into a procedure for predicting satellite motion and planetary orbits.

In the related Satellite Chapter 2 on Kepler Orbits, we discuss how to write a working computer program we call ***Orbit 1*** for calculating the orbits. We used this program to create the plots shown in Figures (21) and (22). The fact that these plots obeyed Kepler's first two laws of planetary motion provides fairly good evidence that the program Orbit 1 works correctly.

The advantage of having the Orbit 1 program is that it can serve as a laboratory for studying orbital motion. The program is based on Newton's law of gravity which means that we are seeing the planetary orbits that result from a $1/R^2$ gravitational force law. And with the program we are then able to study what is special about the $1/R^2$ force. There are two important results we will discuss here.

In Figures (21) and (22) we ran the Orbit 1 program long enough to plot one orbit of the planet. In Satellite Chapter 2, we let the program continue for just under 13 orbits and got the results shown in Figure (24). The point is that the planet kept retracing precisely the same orbit, time after time. That is a special feature of the $1/R^2$ force law, the resulting elliptical orbits are stable.

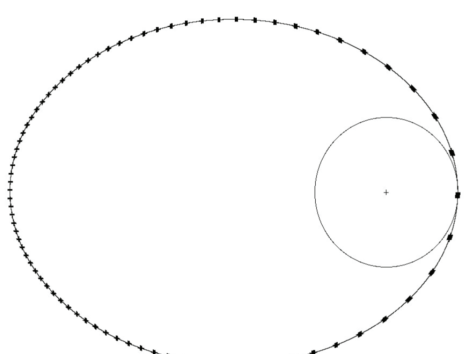


Figure 24
For this plot, the planet went around nearly 13 times, in precisely the same orbit.

In Figure (25) we modified the gravitational force law to

$$F_g = GM_{sun}m_{planet}/R^{1.9}$$

In other words we changed the R dependence of the force from $1/R^2$ to $1/R^{1.9}$. This small change in the exponent of R produced a significant change in the predicted motion of the planet. The ellipse of the orbit starts rotating around the sun. This is called a ***precession*** of the ellipse.

The precession of the elliptical orbit of a planet provides an extremely sensitive test of the theory of gravity. You can see that a change of the exponent from 2 to 1.9 sends the ellipse spinning. Even a tiny change in the exponent will eventually show up if we let the program run long enough. This leads us to a discussion of Einstein's theory of gravity called ***General Relativity***.

After developing the special theory of relativity, Einstein took a look at Newton's theory of gravity and saw that it was not consistent with the principle of relativity. In particular, the Newtonian gravitational force is supposed to point to the instantaneous position of a mass. If I have two masses separated by a distance R, and I suddenly move the first mass, Newton's gravitational force on the second mass should change instantaneously, to follow the new position of the first mass. This should be true no matter how far apart the masses are. This would allow me to send a signal faster than the speed of light, which would violate one of the main consequences of the special theory of relativity.

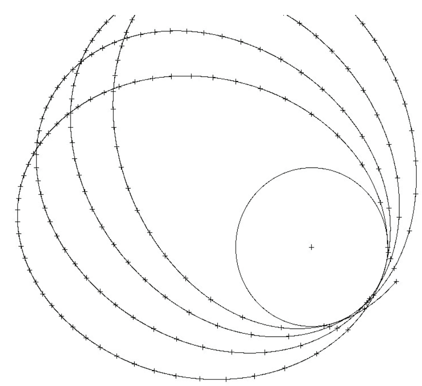


Figure 25
Change the gravitational force from $1/R^2$ to $1/R^{1.9}$ and the orbit and the orbit starts to precess.

Einstein spent the years from 1905, when he presented the special theory of relativity, to 1915, developing his relativistic theory of gravity. The main problem he had testing his relativistic theory was that Newton's law of gravity works so well. Astronomers had spent 250 years applying Newton's laws to planetary motions, and found almost no case where the laws failed. In fact the one clear case involved the elliptical orbit of the planet Mercury.

Most of the planetary orbits are so nearly circular that it is difficult to detect any precession of the ellipse. Mercury's orbit is sufficiently elliptical that the precession of the ellipse can be accurately measured.

Even if Newton's law of gravity were exactly right, Mercury's ellipse should still precess due to the gravitational force of the other planets acting on Mercury. This precession is very small, slightly less than .2 degrees ***per century***. But this small number is accurately measured because astronomers have been observing Mercury's orbit for 3,000 years, or 30 centuries. Over this time Mercury's ellipse has precessed 30 × .2 degrees = 5 degrees, which is easily observed.

When measuring small angles, astronomers divide the degree into ***60 minutes of arc*** and for even smaller angles, divide the minute of arc into ***60 seconds of arc***. One second of arc, 1/3600 of a degree is a very small angle. A basketball 50 kilometers distant, subtends an angle of about one second of arc. In these units, Mercury's orbit precesses about 650 seconds of arc per century.

By 1900, astronomers doing Newtonian mechanics calculations could account for all but 43 seconds of precession of Mercury's orbit, as being caused by the influence of neighboring planets. The 43 seconds of arc discrepancy could not be explained.

One of the most important predictions of Einstein's relativistic theory of gravity is that it predicts a 43 seconds of arc per century precession of Mercury's orbit, a precession caused by a change in the gravitational force law and not due to neighboring planets. Einstein used this explanation of the 43 seconds of arc discrepancy as the experimental foundation for his relativistic theory of gravity.

(To get the 43 seconds of arc per century precession of Mercury's orbit, using a modified gravitational force law, the force would have to be proportional to $1/R^{2.00000016}$ instead of $1/R^2$.)

Einstein's theory, General Relativity, also predicts that starlight passing close to the surface of the sun will be deflected due to a gravitational force acting on the light. Ordinarily you cannot see stars close to the sun, but you can during an eclipse of the sun. If starlight is deflected by the sun, then during an eclipse, a star near the sun will appear to be just slightly out of position. In 1917 the famous English astronomer Sir Authur Eddington led an eclipse expedition to measure the deflection, and confirmed this prediction of Einstein's theory.

APPENDIX

PLANETARY UNITS

When studying the motion of earth satellites, we are faced with large exponents in quantities like the earth mass and the gravitational constant G which are 5.98×10^{24}kg and $6.67 \times 10^{-11} m^3/kg\ sec^2$ respectively. The calculations we have done so far using these numbers have required a calculator, and we have had to work hard to gain insight from the results.

Here we introduce a new set of units, which we will call *planetary units*, that makes satellite calculations much simpler and more intuitive. To design a new set of units we must first decide what will be our unit mass, our unit length, and our unit time, and then work out all the conversion factors so that we can convert a problem into our new units.

For working earth satellite problems, we have found that it is convenient to take the earth mass as the unit mass, the earth radius as the unit length, and the hour as the unit time.

$$m_{earth} = 1 \quad \textit{earth mass}$$

$$R_{earth} = 1 \quad \textit{earth radius}$$

$$hour = 1$$

With these choices, speed, for example, is measured in (earth radii)/ hr, etc.

This system of units has a number of advantages. We can set m_e and r_e equal to 1 in the gravitational force formulas, greatly simplifying the results. We know immediately that a satellite has crashed if its orbital radius becomes less than 1. Typical satellite periods are a few hours and typical satellite speeds are from 1 to 10 earth radii per hour. What may be a bit surprising is that both the acceleration due to gravity at the surface of the earth, g, and Newton's universal gravitational constant G, have the same numerical value of 20.0.

Table 1, on the next page, shows the conversion from MKS to planetary units of common quantities encountered in the study of satellites moving in the vicinity of the earth and the moon.

Exercise 8

We will have you convert Newton's universal gravitational constant G into planetary units. Start with

$$G = 6.67 \times 10^{-11} \frac{meters^3}{kg\ sec^2}$$

Then multiply or divide by the conversion factors

$$3600 \frac{sec}{hr}$$

$$5.98 \times 10^{24} \frac{kg}{earth\ mass}$$

$$6.37 \times 10^{6} \frac{meters}{earth\ radii}$$

until all the dimensions in the formula for G are converted to planetary units. (I.e., convert from seconds to hours, kg to earth mass, and meters to earth radii.) If you do the conversion correctly, you should get the result

$$G = 20 \frac{(earth\ radii)^3}{(earth\ mass)\ hr^2}$$

Exercise 9

Explain why g and G have the same numerical value in planetary units.

As an advertisement for how easy it is to use planetary units in satellite calculation, let us calculate the period of Sputnik 1, the first artificial satellite. Sputnik was so low (less than 200 km up) that its orbital radius was nearly equal to the earth's radius R_{earth}, and its acceleration toward the center of the earth was essentially the same as the projectiles we studied in the introductory lab, namely g_e.

Using the formula

$$a = \frac{v^2}{r} = g_e$$

we get

$$g_e = \frac{v^2_{Sputnik}}{R_{earth}} = \frac{v^2_{Sputnik}}{1}$$

Therefore

$$v_{Sputnik} = \sqrt{g_e} = \sqrt{20}\ \frac{earth\ radii}{hr}$$

Now the satellite travels a total distance $2\pi r_e$ to go one orbit, therefore the time it takes is

$$\text{Sputnik period} = \frac{2\pi r_e}{v_{Sputnik}} = \frac{2\pi}{\sqrt{20}} = 1.4 \text{ hrs}$$

Compare the algebra that we just did with what we went through to calculate the period of Glenn's orbit (Equation 25), and you can see how much easier it is to use planetary units for earth satellite calculations. If you have watched satellite launches on television, you may recall waiting about an hour and a half before the satellite returned.

Exercise 10

A satellite is placed in a circular orbit whose radius is $2r_e$ (it is one earth radii above the surface of the earth).

(a) What is the acceleration due to gravity at this altitude?

(b) What is the period of this satellite's orbit?

(c) What is the shortest possible period any earth satellite can have? Explain your answer.

Table 1 Planetary Units

Constant	Symbol	Planetary units	MKS units
Gravitational Constant	G	20	$6.67 \times 10^{-11} \frac{m^3}{kg\ sec^2}$
Acceleration due to gravity at the earth's surface	g_e	20	$9.8\ m/sec^2$
Earth mass	m_e	1	5.98×10^{24} kg
Moon mass	m_{moon}	.0123	7.36×10^{22} kg
Sun mass	m_{sun}	3.3×10^5	1.99×10^{30}kg
Metric ton	ton	1.67×10^{-22}	1000 kg
Earth radius	r_e	1	6.37×10^6 m
Moon radius	r_{moon}	.2725	1.74×10^6 m
Sun radius	r_{sun}	109	6.96×10^8 m
Earth orbit radius	$r_{earth\ orbit}$	23400	1.50×10^{11} m
Moon orbit radius	$r_{moon\ orbit}$	60	3.82×10^8 m
Hour	hr	1 hr	3600 sec
Moon period	lunar month (siderial)	656 hrs	2.36×10^6 sec (= 27.32 days)
Year	yr	8.78×10^3 hrs	3.16×10^7 sec

CHAPTER 8 REVIEW

Coming into this chapter we had experimental definitions of acceleration (through strobe photographs of Chapter 3) and mass (via the recoil experiments of Chapter 6). We then began this chapter by using Newton's second law $\vec{F} = m\vec{a}$ to introduce the concept of force.

Newton's second law tells us that forces cause acceleration, but you have to use Newton's second law in a very special way. If you have only one force acting on an object, as in the case of the steel ball projectile where only gravity was important, then we could define the gravitational force as simply the ball's mass m times the gravitational acceleration $\vec{g}$ to get the formula $\vec{F}_g = m\vec{g}$.

In this case of the Styrofoam projectile, air resistance was important and we had to deal with two forces $\vec{F}_g$ and $\vec{F}_{air}$ acting at the same time. The way to use Newton's second law in this more general case is to first add up the acting forces to get the net force

$$\vec{F}_{net} = \vec{F}_g + \vec{F}_{air} \tag{32}$$

and then use Newton's second law

$$\vec{F}_{net} = m\vec{a} \tag{33}$$

To check this approach, solve for $\vec{a}$ to get

$$\vec{a} = \frac{\vec{F}_{net}}{m} = \frac{\vec{F}_g}{m} + \frac{\vec{F}_{air}}{m} \tag{34}$$

Using $\vec{F}_g/m = \vec{g}$ and $\vec{F}_{air}/m = \vec{a}_{air}$ we get

$$\vec{a} = \vec{g} + \vec{a}_{air} \tag{35}$$

which is the experimental result we got from the strobe analysis in Figure (4) and back in Figure (3-28).

This led to the general rule for applying Newton's second law. First, find all the forces acting on an object. Then take the vector sum of the forces to get the net force $\vec{F}_{net}$. Finally, the objects acceleration is then given by $\vec{a} = \vec{F}_{net}/m$.

After Chapter 5, in Essay 1, we described Newton's step-by-step method for predicting the motion of an object whose acceleration vector $\vec{a}$ was known. We applied this to the example of projectile motion with air resistance with $\vec{a} = \vec{g} - k\vec{v}$. In the related Satellite Chapter 1 we showed you how to write a computer program that carried out Newton's step-by-step method.

With Newton's second law, we have a general procedure for calculating an object's acceleration $\vec{a}$, if we know all the forces acting on the object. In Essay 2 following this chapter we discuss various examples of using Newton's step-by-step method for more complex problems. The examples include Kepler orbits, a Kepler orbit with a modified gravitational force law, and satellite reentry as the satellite is slowed by air resistance.

The latter part of this chapter deals with Newton's universal law of gravitation. By showing that the same law applied to an apple falling to the ground and to the moon orbiting the earth, Newton demonstrated that "heavenly" objects like the moon obeyed the same laws of physics as objects down here on earth.

One of the main features of Newton's law of gravity is that the gravitational force between two objects falls off as the square of the separation R between the objects. This $1/R^2$ dependence of gravitational forces can be determined from Kepler's three laws of planetary motion.

The three laws are (1) the planets move in elliptical orbits with the sun at one focus, (2) the planets radius vector $\vec{R}$ sweeps out equal areas in equal times, and (3) the ratio r^3/T^2 is the same force for all planets, r being the orbit's semi major axis and T its period.

With a computer constructed strobe photograph of a planetary orbit, we could check that the orbit obeyed Kepler's first two laws, and we could determine that the planet's acceleration vector always points toward the sun. Using the trick of choosing a strobe time Δt proportional to the radius vector R (Figure (22), we could show that the planet's acceleration dropped off as $1/R^2$. The fact that Kepler's laws imply that the gravitational acceleration drops off as $1/R^2$ was one of the key pieces of information Newton had in proposing his law of gravity.

CHAPTER EXERCISES

Exercise 1 0n page 7

Explain how you would calibrate a set of bathroom scales if your laboratory were on the surface of the moon.

Exercise 2 0n page 8

Combine Newton's second law with his law of gravity and show that the dimensions for G are correct.

Exercise 3 0n page 10

a) Calculate the magnitude of the moon's acceleration a_{moon} toward the center of the earth.

b) Calculate the ratio a_{apple}/a_{moon}.

Exercise 4 0n page 12

The density of water is 1 $gram/cm^3$. The average density of the earth's outer crust is about 3 times as great. Use Cavendish's result for the mass of the earth to decide if the entire earth is like the crust.

Exercise 5 0n page 12

Calculate the upward acceleration of the earth when a student jumps off a diving board.

Exercise 6 0n page 15

Calculate the height of a geosynchronous satellite.

Exercise 7 0n page 19

Using the computer plot of a satellite in a Kepler orbit, show that the satellite obeys Kepler's first two laws of planetary motion.

Exercises with Planetary Units

Exercise 8 0n page 25

Calculate G in Planetary Units.

Exercise 9 0n page 25

Explain why g and G have the same numerical value in planetary units.

Exercise 10 0n page 26

A satellite is placed in a circular orbit whose radius is $2r_e$ (it is one earth radii above the surface of the earth).

(a) What is the acceleration due to gravity at this altitude?

(b) What is the period of this satellite's orbit?

(c) What is the shortest possible period any earth satellite can have?

REVIEW QUESTIONS & EXERCISES

(Try doing these exercises without looking back at the chapter. But look back if you get stuck.)

11) Use Newton's law of gravity and Newton's second law to show that Newton's universal constant G has the dimensions of $\text{meter}^3/\text{kg sec}^2$.

12) Derive the formula for the magnitude of the acceleration g due to gravity at the surface of the earth using Newton's universal law of gravitation. What part of this calculation was an incentive for Newton to invent calculus?

13) Describe how the mass of the earth was first measured.

14) How much mass do you have to add to the earth to increase g from 9.8 $meters/sec^2$ up to 10 $meters/sec^2$. (In order to simplify physics homework problems?). Assume the earth radius is unchanged.

15) (An advanced question that requires some graphical work.) Show that Kepler's second law is a direct consequence of the conservation of the planet's angular momentum about the sun.

16) After you jump off a diving board but before you hit the water, what is the magnitude and direction of the gravitational force you are exerting on the earth?

17) If a satellite is in free fall, why doesn't it hit the earth?